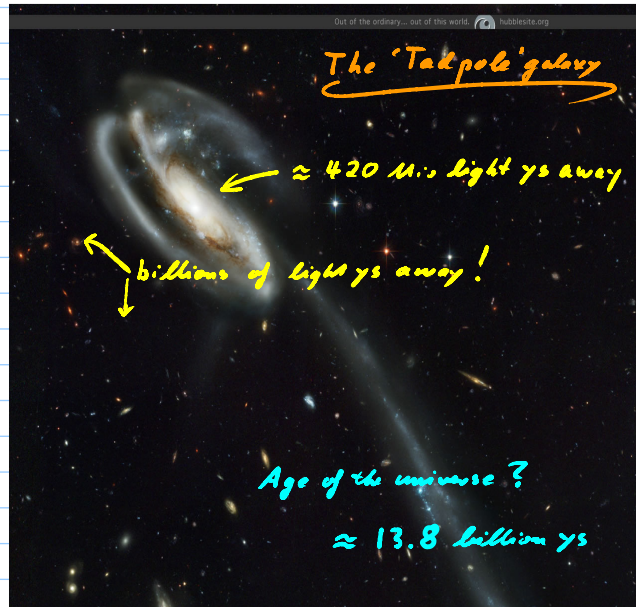


[www.math.uwaterloo.ca/~akempf/AMATH875-F13.shtml](http://www.math.uwaterloo.ca/~akempf/AMATH875-F13.shtml)

## Review: From special to general relativity

*This lecture is not taught.*



## Part 1: Special Relativity

Q: What is the Principle of Relativity?

A: The laws of nature take the same mathematical form for each non-accelerated observer (i.e. for all inertial observers)

Implications?

- Even when we travel very fast, our biological functions are normal.
- (It was thought that travel sickness had to do with velocity itself, rather than the shaking)

□ All velocities are relative :

being fast or slow only makes sense relative to another object.

*Note:* Accelerations though are not relative to other objects - we feel directly if we are accelerated or not. In this context, later the 'Principle of Equivalence' will come in and lead to G.R.

□ The speed of light,  $c$ , is the same for all inertial observers.

Q: Why?

A: 1) The Maxwell equations of Electromagnetism allow one to calculate  $c$ .

2) By the Principle of Relativity the Maxwell equations are exactly the same for all inertial observers

3) Thus, each inertial observer finds the same value for  $c$ .

□ Similarly, all inertial observers

find the same values for all numbers

that are implied by laws of nature,

e.g., the same

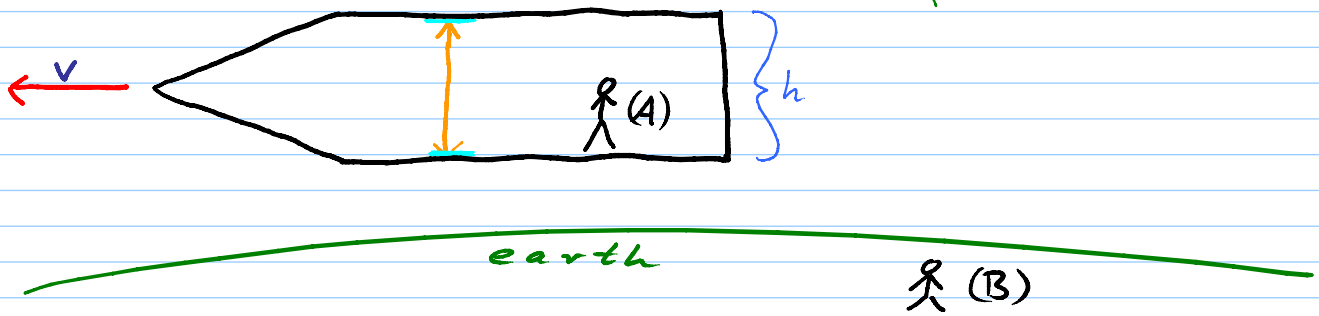
◦ charges, total spin and rest masses of all elementary particles

◦ speed of gluons ( $c$ )

□ Crucially: Temporal and spatial distances are relative!

# Review: Time dilatation

Q: Why bother with review?  
A: Illuminates role of coordinate systems → crucial insight for GR



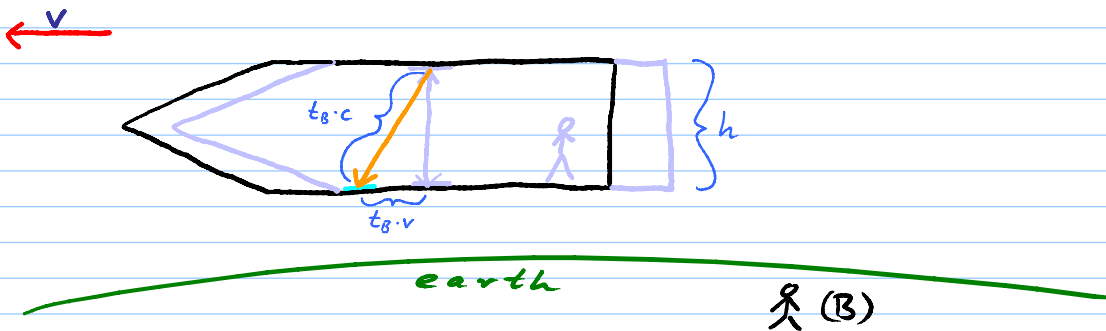
Observer A uses as his clock a light ray that oscillates between mirrors on the ceiling and floor of his rocket.

For observer A, how long does it take for the light ray to go from say ceiling to floor?

$$t_A = \frac{h}{c} \quad (1)$$

For observer B, how long does it take for the same light ray to go from ceiling to floor?

Note: Observer B agrees with observer A on the value of  $h$ : If not, then e.g. head-on colliding equally built rockets could find out who was fast and who was slow.



⇒ For observer B, the rocket travelled the distance  $t_B \cdot v$  and the light ray travelled the distance  $t_B \cdot c$ . By Pythagoras:

$$t_B^2 \cdot c^2 = t_B^2 \cdot v^2 + h^2 \quad (2)$$

From (1) and (2), eliminate  $h$  ⇒

$$t_B^2 c^2 = t_B^2 v^2 + t_A^2 c^2$$

Thus:

$$t_A = t_B \sqrt{1 - \frac{v^2}{c^2}} \quad (3)$$

$\leq 1$

So, time passes slower for observer A than for observer B?

Yes, but:

By principle of relativity must also have that time passes slower for B than for A.

Q: How can this be?

A: Need to be very clear about the role of coordinate systems!

## Coordinate systems

How to make one?

- Buy a lot of clocks and a lot of metal rods of equal length.
- Use the rods to build a rectangular lattice

□ Place clocks at all the vertices of the lattice.

□ Make sure the clocks are synchronized, e.g. by having light rays oscillate in between neighboring clocks.

### Resolution of above paradox:

□ A's wrist watch travels past many clocks of B's coordinate system. We calculated above that A's wrist watch will lag behind the successive readings of the clocks of B that it passes by.

□ Analogously, B's wrist watch travels past many clocks of A's coordinate system - and lags behind them!

→ The situation is entirely symmetric between A and B, as it must be by the principle of relativity.

## The twin paradox ... and the need for GR

Consider twins:

- one, A, travels and comes back
- one, B, stays behind.

As they meet and shake hands they notice:  
A has aged less.

Why?

- B was always inertial  $\Rightarrow$
- by principle of relativity, for B the laws of nature always took the usual form  $\Rightarrow$

- B can use above calculation and concludes that A's wrist watch lags behind his coordinate system's clocks that it passes.
- Since only one can be right when they finally meet we conclude:

Observer A cannot always apply the above calculation to find that B will be younger.

- $\Rightarrow$  For A, the laws of nature take a new, nontrivial, form while he is accelerating, so that above calculation fails.

Note: It seemed a small problem because can always choose to work in an inertial coordinate system - and in it the laws of nature take the usual form.

But: Can we really always build an inertial coordinate system? No, because of gravity!

→ Einstein:

The laws of nature should take the same mathematical form in all coordinate systems (i.e. not only in inertial ones).

He found:

This is possible, if gravity is taken into account.

His key insight:

The principle of equivalence.

## Part 2: Motivation for GR

What had Einstein achieved with special relativity?

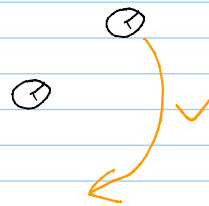
- ▣ Laws of nature took the same form in all inertial, cartesian coordinate systems (i.e. in all coordinate systems that are a freely moving rectangular arrangement of equal length rods with synchronized clocks at the vertices).
- ▣ He could deduce the form of the laws of nature in an arbitrary coordinate system by writing them down in an inertial cartesian

system and by then simply transforming the variables to those of the arbitrary system.

**E.g.:** A straight path in an inertial cds will look bent in an arb. cds.

⇒ Laws of nature in generic coordinate systems include pseudo forces

**E.g.:** In an inertial cds., consider a clock at rest and a clock circling it:



The circling clock is fast, thus keeps

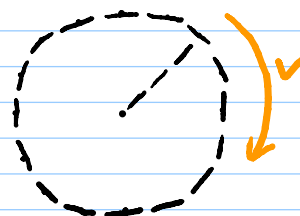
lagging behind the clock at the centre.

But: In a suitably rotating cds, both clocks are at rest!

⇒ In generic cds, even clocks at rest may not be synchronizable.

(Recall that as the travelling twin accelerates he cannot keep his cds's clocks synchronized)

**E.g.:** Consider, built from equally-made rods, in an inertial cartesian cds:

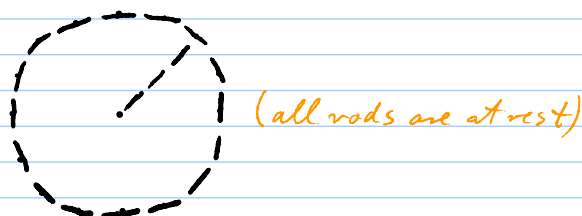


$$\frac{\text{Circumference}}{\text{Diameter}} = \pi$$

→ The rods of the circumference are fast, i.e. shorter, i.e. more than usual fit around.

→ The radial rods get thinner but stay the same length. Thus, as many as usual fit on the radius.

→ But now consider this in a rotating cds in which all rods are at rest.



In this cds:

$$\frac{\text{Circumference}}{\text{Diameter}} > \pi$$

→ Einstein noticed:

If the laws of nature were to take the same mathematical form in all cds, they would

- contain elements of non-euclidean geometry
- allow for cds with non-synchronizable clocks
- describe forces that at least include the

pseudo-forces. (And thus, via the equiv. principle, expect also gravity to enter in more general situations, as we will see soon!)

His next moves were ingenious:

1. In order to improve on special relativity, he asked where the limits of its validity are, and found:

There are no large inertial cartesian cds!

### Reason:

Assume a rest. sys. of rods and clocks is large enough to contain stars. Since it is rigid, if one part of it is inertial, i.e. freely falling, other parts of it are not inertial i.e. not freely falling, e.g. because a star may be pulling them in some other direction.

2. Rather than seeing this as the complete downfall of special relativity, Einstein conjectured:

Even in the presence of gravity, an inertial, cartesian coordinate system can be constructed around

every point (event), with arbitrary precision, at least in a suitably small neighborhood of that point. And, the laws of special relativity hold in that local coordinate system.

### Note:

Recalling that all bodies fall equally in gravity (equivalence of inertial and gravitational mass), Einstein therefore postulated more generally the;

## "Equivalence Principle":

The laws of nature (and therefore also e.g. all biol. processes) are locally the same, whether one is freely floating in empty space or whether one is freely falling under the influence of gravity.

Note: Accelerations due to say electric attraction could not be included:  
for example, pos. and neg. charges are pulled differently.

3. Einstein remembered a similar problem earlier faced by Riemann:

- Euclidean geometry is nice and simple, but how to describe a smooth but curved manifold?
- Riemann solved the problem after observing that at each point such manifolds are flat to any arbitrary precision - within a sufficiently small neighborhood.

⇒ Einstein's strategy:

Develop 3+1 dim Riemannian geometry, in which, locally, the geometry of spec. relativity holds.