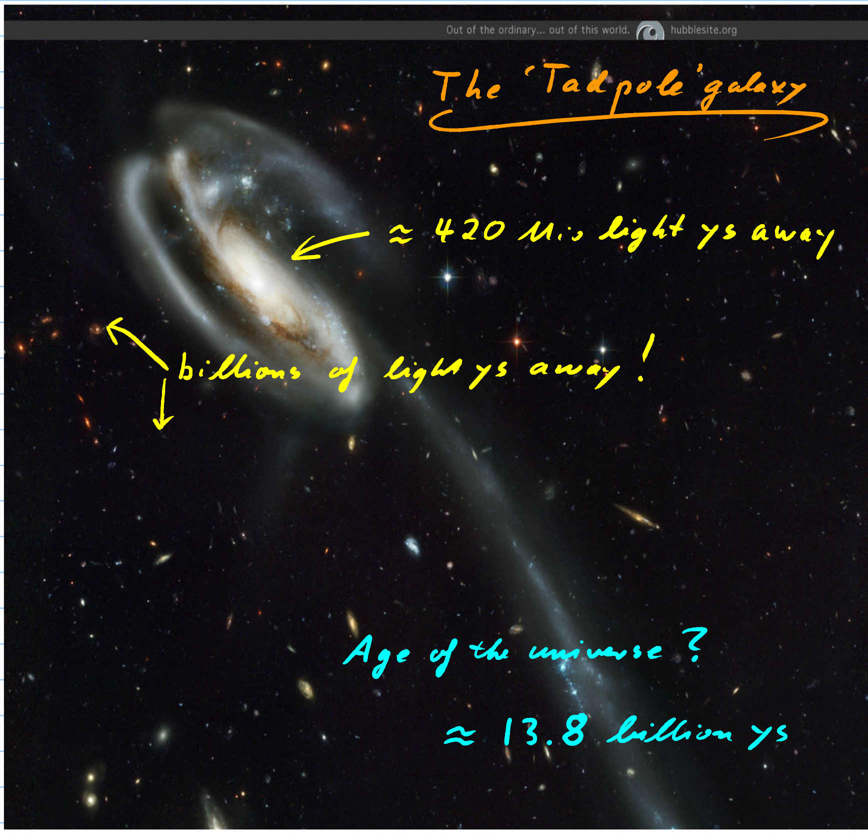


www.math.uwaterloo.ca/~akempf/AMATH875-F13.shtml

Spacetime's curvature can be seen directly:

HST: ABELL2218

How to describe spacetime?

A. Math

Strategy: ▢ Start with a mere "set" of points (events), $\mathcal{M}$

Then add structure:

▢ Define open neighborhoods (i.e., a "topology" on $\mathcal{M}$)

▢ Define "separability" of points (i.e. Hausdorf condition)

▢ Define "continuity" (preimage of open sets is open)

▢ Define "differentiability" (via chart change diffability)

later: ▢ Define tangent & tensor spaces

⋮

$\boxed{\text{Curvature = nontriviality of parallel transport}}$

Other descriptions of curvature?

(Why consider others? May be useful for quantum gravity b/c what's on previous page is likely over idealized.)

▢ Curvature = sum of angles in triangle $\neq \pi$

▢ Curvature = nontriviality of Pythagoras law

▢ Curvature = tidal forces. Math of it: Sectional curvatures

▢ Curvature $\stackrel{?}{=}$ nontrivial sound of object when vibrating

This field is called Spectral Geometry.

Interesting b/c connects mathematical languages of quantum theory (spectra etc) and general relativity.

We'll cover it to show recent developments: See arxiv 1212.5297 (PRL)

▢ Curvature $\stackrel{?}{=}$ nontrivial entanglement in vacuum fluctuations

See arxiv: 1302.3680 (Found. of Physics)

B) Structure and properties of General Relativity?

- Equations of motion
  for scalars, vectors, spinors and curvature
- Symmetries
  local and global conservation laws, if any!
- Tetrad formulation, GR as a gauge theory
- Singularities, and their unavoidability

C) Applications to Cosmology

- Classification of exact solutions
- Models of cosmological matter
- FRW models, while
  using the tetrad formalism
  to exercise it. (e.g. for later use in quantum gravity)
- Cosmic inflation
- Black holes

Pseudo-Riemannian Differential Geometry

Differentiable Manifolds

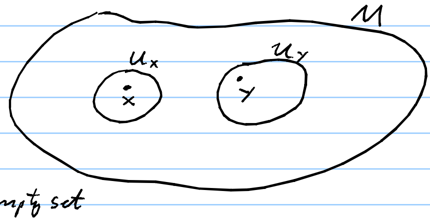
(Riemann ≈ 1850s, Poincaré ≈ 1890s, Whitney ≈ 1930s ...)

Def: An n-dimensional topological Manifold, M, is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Here:

Def: A topological space, M, is a set, together with a specification of subsets U_i, which will be called "open subsets", which must obey $U_i \cap U_j$ is open, and $\bigcup_\lambda U_\lambda$ is open.

Def: A topological space M is called Hausdorff, if it is separable, i.e., if $x, y \in M$ and $x \neq y$ then x, y are elements of some disjoint open sets.



$\forall x, y: x \neq y \;\exists\, U_x, U_y$ open : $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \{\}$

↑ "for all" ↖ "there exist" ↑ empty set

Notice: Now M has a topology consisting of open sets. And, of course, $\mathbb{R}^n$ also does.

Recall: If A, B are topol. spaces, then $f: A \to B$ is called continuous if $\forall V \subset A$, $U := f(V)$: (U open $\Rightarrow$ V open)

→ We can now express the idea that M is continuously parametrizable:

Def: M is called locally homeomorphic to $\mathbb{R}^n$, if each point, p, has a neighborhood $U(p)$, and an invertible continuous map $h: U(p) \to \mathbb{R}^n$.

Def: A local homeomorphism,

$$h: \underset{\text{called "domain"}}{U} \to \mathbb{R}^n, \quad U \subset M$$

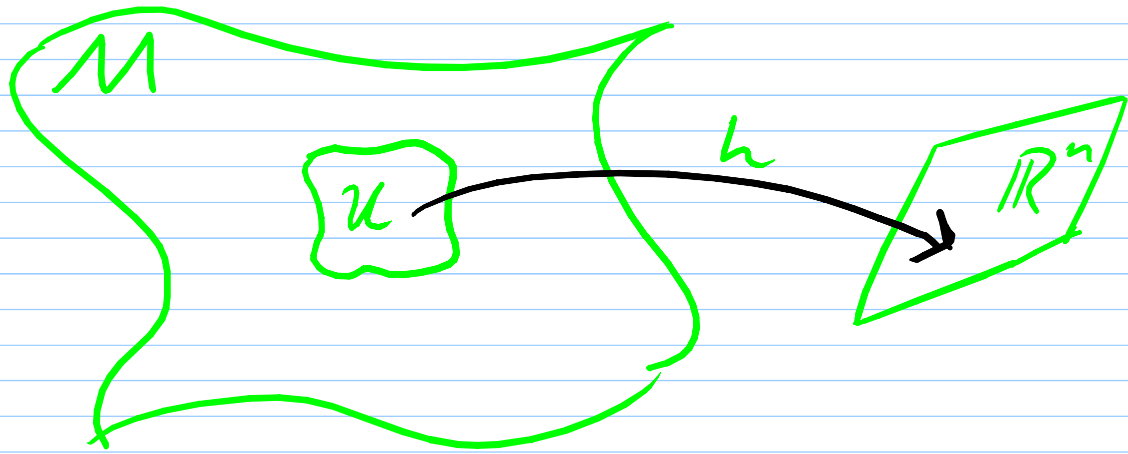
is called a chart of M.

For any point $q \in U$ its image

$$h(q) \in \mathbb{R}^n$$

is a set of n numbers $(x_1, x_2, \ldots, x_n)$ called the coordinates of q.

Def: A chart, h, with domain $\mathcal{U}$,



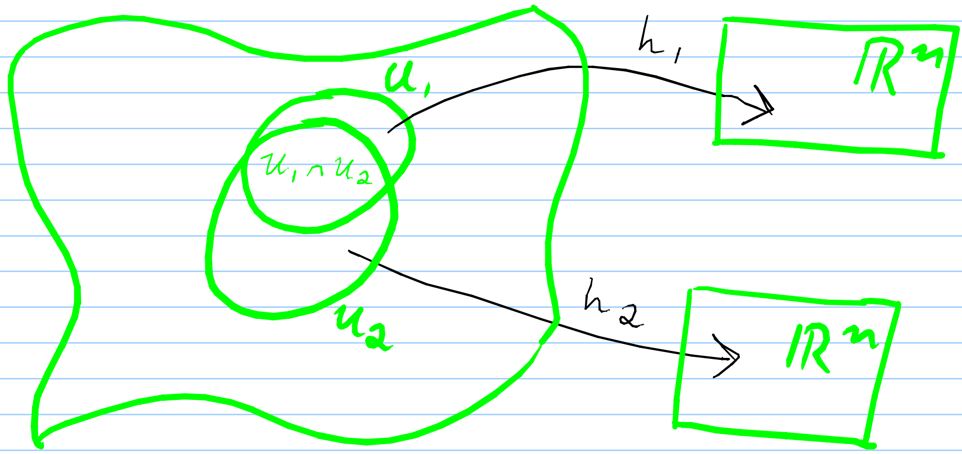
is also called a

local coordinate system for $\mathcal{U}$.

Def: A collection of charts h_α with domains $\mathcal{U}_\alpha$ is called an atlas if $\bigcup_\alpha \mathcal{U}_\alpha = M$.

→ What, if we want to change coordinates, i.e. if we want to re-label the points of (e.g. a subset of) the manifold?

Consider 2 charts h_1, h_2, with intersecting domains $U_1 \cap U_2 \neq \emptyset$:



Then, $h_{12} = h_2 \circ h_1^{-1}$ is a continuous change of coordinates map $h_{12} : \mathbb{R}^n \to \mathbb{R}^n$.

Notice: For maps $\mathbb{R}^n \to \mathbb{R}^n$ we know what differentiability means!

Strategy: Let us define the diff'ability of an atlas through the diff'ability of its chart changes:

Def: An atlas is called C^r differentiable, if all its coordinate changes, $h_{\alpha\beta}$, are C^r diffeomorphisms, i.e., r times continuous differentiable.

Strategy: Enlarge atlas so every point of $\mathcal{M}$ is in multiple charts. Then, diffability of $\mathcal{M}$ is definable through atlas diffability

Def: Given a C^r differentiable atlas, $\mathcal{A}$, we can generate a maximal C^r differentiable atlas, $D(\mathcal{A})$, by adding all charts whose chart changes with charts in $\mathcal{A}$ are differentiable.

Def: $D(\mathcal{A})$ is also called a "Differentiable Structure" of class C^r for $\mathcal{M}$.

Def: A differentiable manifold of class C^r is a topol. manifold with a maximal atlas of class C^r, i.e., with a differentiable structure of class C^r.

Theorem: (Whitney)

Every C^k structure with $k \geq 1$ is C^k equivalent to a C^∞ structure (i.e. there is always a suitable set of charts).

I.e. any diffable structure can be smoothened. Any lack of higher diffability is due to unlucky choice of chart.

Def: Since any C^1 manifold is also a C^∞ manifold, we also call diffable manifolds simply smooth manifolds.