

Recall:

E.g.: $L_{EM} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$



$$S[g_{\mu\nu}, \Psi] = \int \left(\frac{1}{16\pi G} R(g_{\mu\nu}(x)) + L_{\text{matter}}(g_{\mu\nu}(x), \Psi^{(i)}(x), \Psi^{(i)}_{;\mu}(x)) \right) \sqrt{g} d^4x$$

$$\frac{\delta S'}{\delta \Psi^{(i)}} = 0 \quad \Rightarrow \quad \text{Eqs. of motion of matter}$$

(Maxwell, Klein Gordon eqns. etc)

$$\frac{\delta S'}{\delta g_{\mu\nu}} = 0 \quad \Rightarrow \quad \text{Einstein equations:}$$

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}$$

What if we allow ourselves arbitrary bases in $T(\mathcal{M})$?

Recall:

□ Frames $\{\theta^\mu\}, \{e_\mu\}$:

Often, one uses as the bases of $T_p(\mathcal{M})$, and $T_p(\mathcal{M})'$ the canonical bases $\{dx^\mu\}$ and $\{\frac{\partial}{\partial x^\mu}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, \dots, x^3)$. Thus, when changing coordinate system, $x \rightarrow \bar{x}$, one also usually automatically changes basis in $T_p(\mathcal{M}), T_p(\mathcal{M})'$.

Important: The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-) tangent spaces, namely from one canonical basis to another canonical basis, when we change coord. system.

Recall:
 (a fixed vector has different coefficients in different bases):
 $\xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \frac{\partial \bar{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \bar{x}^\nu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu} \Rightarrow \bar{\xi}^\nu = \frac{\partial \bar{x}^\nu}{\partial x^\mu} \xi^\mu$

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu}$$

We notice: If we choose a fixed basis, say $\{\theta^\mu\}, \{e_\mu\}$ then the coefficients of tensors no longer depend on the choice of coordinates!

E.g.: $\xi = \tilde{\xi}^\mu e_\mu$ ← the same numbers in every coordinate system.

Conversely: Even staying with one coordinate system, we can freely change our choice of basis in the (co-) tangent spaces:

$$\begin{aligned} \theta'^\mu &= A^\mu{}_\nu \theta^\nu \\ e'_\mu &= (A^{-1})^\nu{}_\mu e_\nu \end{aligned}$$

← scalar functions.

So we have e.g.:

$$\xi = \xi^\mu e_\mu = \xi^\mu A^\nu{}_\mu e'_\nu = \xi'^\nu e'_\nu$$

I.e.: $\xi'^\nu = A^\nu{}_\mu \xi^\mu$

Examples: □ The curvature form: $\Omega'^\mu{}_\nu = A^\mu{}_\alpha (A^{-1})^\beta{}_\nu \Omega^{\alpha}{}_{\beta}$

□ But: the connection form $\omega'^\mu{}_\nu(\xi) = \xi^k \Gamma'^\mu{}_{\kappa\nu}$ obeys:

$$\omega'^\mu{}_\nu = A^\mu{}_\alpha \omega^{\alpha}{}_{\beta} (A^{-1})^\beta{}_\nu - (dA)^\mu{}_\alpha (A^{-1})^\alpha{}_\nu$$

How to specify frames?

In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:

$$\theta^i(x) = A^i_j(x) dx^j$$

(Another possibility? Take n scalar functions $f^1, \dots, f^n$ and define $\theta^i := df^{(i)}$. For generic functions these $\{\theta^i\}$ will be linearly independent almost everywhere)

Note: the $A^i_j(x)$ change nontrivially when changing the coordinate system!

Our choice now: orthonormal frames, or "Tetrads":

□ We say that a frame $\{\theta^r\}, \{e_r\}$ is orthonormal if in this frame, for all $p \in M$:

$$g(e_r, e_s) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)_{r,s} = \eta_{rs} \quad \text{i.e. if: } g = -\theta^0 \otimes \theta^0 + \sum_{(i,j)} \theta^i \otimes \theta^j$$

□ Existence? Always: At each $p \in M$ may choose e.g. $\theta^r = dx^r$ where dx^r are canonical ON basis at centre of a geodesic cds.

□ Uniqueness?

For a given space-time, (M, g) , any ON frame yields a new ON frame by transforming the bases through

$$\theta'^r(x) = \Lambda(x)^r_v \theta^v(x),$$

if the linear maps $\Lambda(x)$ preserve the orthonormality:

$$\eta_{rs} \theta'^r \otimes \theta'^s = \eta_{ab} \theta^a \otimes \theta^b$$

i.e. if: $\Lambda^r_a \Lambda^s_b \eta_{rs} = \eta_{ab}$ recall: this is the defining equation for Lorentz transformations. (*)

$\Rightarrow$ Frames are unique up to local Lorentz transformations.

Re-express the degrees of freedom:

- We used to specify space-times through these data: $(\mathcal{M}, g)$
- Now, let us specify space-times, **equivalently**, through data $(\mathcal{M}, \{\theta^i\})$:

Namely:

Assume the $\{\theta^i\}$ are given w. resp. to a basis $\{dx^\mu\}$ through functions A^μ_ν ,

$$\theta^\mu(x) = A^\mu_\nu(x) dx^\nu$$

so that: $g_{\mu\nu} = (\theta^i, \theta^j) = \eta_{ij}$ in the basis $\{\theta^i\}$!

Notice: knowing the $A^\mu_\nu(x)$, we can reconstruct $g_{\mu\nu}(x)$ in basis $\{dx^\mu\}$:

We use that the abstract g is the same in every basis:

$$g = \underbrace{\eta_{ij}}_{\text{because it's tetrad}} \theta^i \otimes \theta^j = \eta_{\mu\nu} \overbrace{A^\mu_a A^\nu_b}^{= g_{ab}(x)} dx^a \otimes dx^b = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$$

$\Rightarrow$

$$g_{ab}(x) = \eta_{\mu\nu} A^\mu_a(x) A^\nu_b(x)$$

$\Rightarrow$

$\{\theta^i(x)\}$ indeed determines $g_{\mu\nu}(x)$:

$\Rightarrow$

The $A^\mu_\nu(x)$ carry all physical (here shape) info!

But how exactly does $A^i_j(x)$ encode $C^i_{jk}, \omega^i_j, \Omega^i_j$?

□ Start with orthonormal frame: $\Theta^i(x) = A^i_j(x) dx^j$

1.) How do the $A^i_j(x)$ determine the $C^i_{jk}(x)$?

Recall from lecture 11:

$$d\Theta^i(x) = -\frac{1}{2} C^i_{jk}(x) \Theta^j(x) \wedge \Theta^k(x)$$

$$\begin{aligned} \text{Here: } d\Theta^i(x) &= A^i_{j,k}(x) dx^k \wedge dx^j \\ &= -\frac{1}{2} C^i_{ab} \Theta^a \wedge \Theta^b = -\frac{1}{2} C^i_{ab} A^a_k A^b_j dx^k \wedge dx^j \end{aligned}$$

$$\Rightarrow A^i_{j,k} = -\frac{1}{2} C^i_{ab} A^a_k A^b_j$$

$$\Rightarrow C^i_{ab}(x) = -2 A^i_{j,k}(x) (A^{-1}(x))^j_k (A^{-1}(x))^k_b$$

2.) The $C^i_{jk}(x)$ yield the $\Gamma^i_{jk}(x)$ through:

$$\begin{aligned} \Gamma^l_{ki} &:= \frac{1}{2} (C^l_{ki} - g_{is} g^{sj} C^s_{kj} - g_{ks} g^{sj} C^s_{ij}) && \text{(Lecture 11)} \\ &+ \frac{1}{2} g^{lj} (g_{jik} + g_{jki} - g_{kij}) && \leftarrow \text{These all vanish because } g_{\mu\nu} = \delta_{\mu\nu} \text{ now} \end{aligned}$$

Notice: This simplifies for orthonormal frames with $g_{\mu\nu}(x) = \delta_{\mu\nu}$!

3.) The $\Gamma^i_{kj}(x)$ yield the $\omega^i_j(x)$:

$$\omega^i_j(x) := \Gamma^i_{kj}(x) \Theta^k(x)$$

4.) The 2nd structure equation yields:

$$\underline{\Omega^i_j(x) := d\omega^i_j + \omega^i_k \wedge \omega^k_j}$$

Recall important identities: (torsionless case)

□ Structure eqn. I:

$$\Theta^i = D\Theta^i = d\Theta^i + \omega^i_j \wedge \Theta^j = 0$$

□ Structure eqn II:

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$$

↑ (Ordinarily: $\Theta^i = dx^i \Rightarrow d\Theta^i = 0$
and $\omega^i_j \wedge \Theta^j = 0$ is $\Gamma^i_{jk} = \Gamma^i_{kj}$)

□ Bianchi identity I:

$$\Omega^i_j \wedge \Theta^j = 0$$

← (Recall: $R^i_{jkl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^m_{jk}\Gamma^i_{lm} - \Gamma^m_{jl}\Gamma^i_{km}$)

□ Bianchi identity II:

$$D\Omega^i_j = 0$$

↗ (From diffeomorphism invariance)

And, in the case of ON frames:

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

Tetrad formulation of GR:

Consider the action, for now, without cosmological constant and without matter:

$$S'_{\text{grav}} = \frac{1}{16\pi G} \int_B \overset{\text{0-form}}{R} \sqrt{g} d^4x$$

Recall Hodge *:

$$\text{If } v = \frac{1}{p!} v_{i_1 \dots i_p} \Theta^{i_1} \wedge \dots \wedge \Theta^{i_p}$$

$$\text{then } *v = \frac{1}{p!} \sqrt{g} \overset{= \pm 1, \text{ totally anti-symmetric}}{\epsilon_{i_1 \dots i_n}} v^{i_1 \dots i_p} \Theta^{i_{p+1}} \wedge \dots \wedge \Theta^{i_n}$$

$$\text{i.e. } *: \Lambda^p \rightarrow \Lambda^{n-p}$$

Thus:

$$S'_{\text{grav}} = \frac{1}{16\pi G} \int_B \underbrace{*R}_{\text{4-form}}$$

Aim now: Re-express $S'_{\mu\nu}$ in terms of θ^μ and $\Omega^{\mu\nu}$.

□ Define: "capital η " is a $(0,2)$ tensor-valued 2-form

$$H_{\mu\nu} := *(\theta^\mu \wedge \theta^\nu) = \frac{1}{2} \overset{1}{\underset{11}{V}} \epsilon_{\mu\nu\gamma\delta} \theta^\gamma \wedge \theta^\delta$$

$$H_{\mu\nu\gamma} := *(\theta^\mu \wedge \theta^\nu \wedge \theta^\gamma) = \frac{1}{2} \overset{1}{\underset{11}{V}} \epsilon_{\mu\nu\gamma\delta} \theta^\delta$$

↑ a $(0,3)$ tensor-valued 1-form.

□ Proposition:

$$*R = H_{\mu\nu} \wedge \Omega^{\mu\nu} \quad \left(\begin{array}{l} \text{it is a } (0,0) \text{ tensor-valued} \\ 4\text{-form} \end{array} \right)$$

i.e.: $\boxed{S'_{\mu\nu}(\theta^\mu) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu}}$

□ Proof:

Use $\Omega^{\mu\nu} = \frac{1}{2} R^{\mu\nu}{}_{\kappa\lambda} \theta^\kappa \wedge \theta^\lambda \Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2 \cdot 2} \epsilon_{\mu\nu\gamma\delta} R^{\mu\nu}{}_{\kappa\lambda} \underbrace{\theta^\gamma \wedge \theta^\delta \wedge \theta^\kappa \wedge \theta^\lambda}_{\epsilon_{\gamma\delta\kappa\lambda} \theta^\gamma \otimes \theta^\delta \otimes \theta^\kappa \otimes \theta^\lambda}$$

Use also: $\epsilon_{\mu\nu\gamma\delta} \epsilon_{\gamma\delta\kappa\lambda} = 2(\delta_{\mu\kappa} \delta_{\nu\lambda} - \delta_{\mu\lambda} \delta_{\nu\kappa}) \Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{4}{4} R^{\mu\nu}{}_{\mu\nu} \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = *R \quad \checkmark$$

(need later for derivation of the Einstein equation)



□ Proposition: $DH_{\mu\nu} = 0$

constant because ON basis

Recall the "first structure equation": $D\theta^\mu = 0$

□ Proof: $DH_{\mu\nu} = D\left(\frac{1}{2} \overset{1}{\underset{11}{V}} \epsilon_{\mu\nu\gamma\delta} \theta^\gamma \wedge \theta^\delta\right) = \frac{1}{2} \epsilon_{\mu\nu\gamma\delta} (\overset{0}{D}\theta^\gamma \wedge \theta^\delta + \theta^\gamma \wedge \overset{0}{D}\theta^\delta)$

The main proposition:

variation, not co-derivative

Variation of the action with respect to $\delta\theta^\mu(x)$ yields:

i.e., we vary the $A^\mu_\nu(x)$ by local Lorentz transformations

$$\delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + d(\text{something})$$

Stokes:

$$\int_B df = \int_{\partial B} f$$

It implies:

$$16\pi G \delta S'_{\text{grav}} = \int_B \delta\theta^\mu \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + \int_{\partial B} (\text{something})$$

← require variation to vanish at boundary ∂B , so: $= 0$

Definition: The "energy-momentum 1-form" T_μ is defined as the solution to:

$$\delta S_{\text{matter}} =: \int_B \delta\theta^\mu \wedge (*T_\mu)$$

$\Rightarrow$ The equation of motion, i.e., the Einstein equation,

$$\frac{\delta(S_{\text{grav}} + S_{\text{matter}})}{\delta\theta^\mu} = 0$$

becomes:

$$-\frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} = 8\pi G *T_\mu$$

Exercise: add the cosmological constant.

Remark: The Einstein form $G_\mu := G_{\mu\nu} \theta^\nu$ obeys

$$*G_\mu = -\frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma}$$

$\Rightarrow$

$$G_\mu = 8\pi G T_\mu$$

↙ (it is a $(0,1)$ tensor-valued 1-form)

Proof of the main proposition:

$$\delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + d(\text{something})$$

Indeed:

$$\delta(*R) = (\delta H_{\mu\nu}) \wedge \Omega^{\mu\nu} + H_{\mu\nu} \wedge \delta\Omega^{\mu\nu}$$

Consider the first term:

$$\delta H_{\mu\nu} = \delta \overset{\text{const.}}{\frac{1}{2} V_g^\gamma \epsilon_{\mu\nu\sigma} \theta^\sigma} \wedge \theta^\sigma$$

$$= (\delta\theta^\mu) \wedge H_{\mu\nu\sigma} \quad \text{by definition of } H_{\mu\nu\sigma} \text{ above:}$$

$$H_{\mu\nu\sigma} := \frac{1}{2} V_g^\gamma \epsilon_{\mu\nu\sigma} \theta^\sigma$$

$$\Rightarrow \delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + \underbrace{H_{\mu\nu} \wedge \delta\Omega^{\mu\nu}}_{\text{examine this term:}}$$

$$\delta\Omega^{\mu\nu} \overset{\text{2nd structure equation}}{=} \delta(d\omega^{\mu\nu} + \omega^\mu{}_\sigma \wedge \omega^{\sigma\nu})$$

$$= d\delta\omega^{\mu\nu} + \delta\omega^\mu{}_\sigma \wedge \omega^{\sigma\nu} + \omega^\mu{}_\sigma \wedge \delta\omega^{\sigma\nu}$$

$$\Rightarrow H_{\mu\nu} \wedge \delta\Omega^{\mu\nu} = d(H_{\mu\nu} \wedge \delta\omega^{\mu\nu}) - (dH_{\mu\nu}) \wedge \delta\omega^{\mu\nu} + H_{\mu\nu} \wedge \delta\omega^\mu{}_\sigma \wedge \omega^{\sigma\nu} + H_{\mu\nu} \wedge \omega^\mu{}_\sigma \wedge \delta\omega^{\sigma\nu}$$

$$\overset{\text{by Def. of } D}{=} (\delta\omega^{\mu\nu}) \wedge \underbrace{DH_{\mu\nu}}_{\text{recall: } = 0 \text{ by Prop. above.}} + d(H_{\mu\nu} \wedge \delta\omega^{\mu\nu})$$

$\Rightarrow$ Indeed:

$$\delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + d(H_{\mu\nu} \wedge \delta\omega^{\mu\nu}) \quad \checkmark$$

General Relativity as a "gauge theory"

Recall:

$$\int_{\text{man}} (\theta^r) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu} \quad \text{Einstein action}$$

$$-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\mu\nu\rho} = 8\pi G *T_\mu \quad \text{Einstein equation}$$

are now the same in all coordinate systems.

In addition:

They are the same also with any choice of ON bases in the tangent spaces, i.e., we have a local symmetry under:

$$\theta^r(x) \rightarrow \tilde{\theta}^r(x) = A^r_{\sigma}(x) \theta^\sigma(x)$$

The $A^r_{\sigma}(x)$ are local Lorentz transformations.

Upshot: $\square$ We can start with any matter theory that is invariant under global Lorentz transformations and, through general relativity, turn it into a theory that is invariant under local Lorentz transformations.

$\square$ Thereby:

Derivatives become covariant derivatives.

A new field is introduced: gravity's Γ .

$\leadsto$ This is analogous to the gauge principle of particle physics:

$\square$ A global symmetry is "gauged" to become local.

$\square$ Derivatives become covariant derivatives

$\square$ A new field is introduced.

The gauge principle:

Action for a Dirac field (electrons, quarks etc):

$$S'[\psi] = \int \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi d^4x$$

It has a global symmetry:

$$\psi(x) \rightarrow \tilde{\psi}(x) := e^{i\alpha} \psi(x), \text{ i.e., } \bar{\psi}(x) \rightarrow \bar{\tilde{\psi}}(x) = e^{-i\alpha} \bar{\psi}(x)$$

$$\Rightarrow S'[\psi] \rightarrow S'[\tilde{\psi}] = S'[\psi]$$

However, no local symmetry:

$$\psi(x) \rightarrow \tilde{\psi}(x) := e^{i\alpha(x)} \psi(x) \quad \bar{\psi}(x) \rightarrow \bar{\tilde{\psi}}(x) = e^{-i\alpha(x)} \bar{\psi}(x)$$
$$S'[\psi] \rightarrow S'[\tilde{\psi}] \neq S'[\psi] !$$

Gauge principle: Introduce a new field $A_\mu(x)$ that transforms so as to absorb the extra term:

$$S'[\psi, A] := \int \bar{\psi}(x) \underbrace{\left(i \gamma^\mu (\partial_\mu + i A_\mu(x)) - m \right)}_{\text{"covariant derivative"}} \psi(x) d^4x$$

Now under

$$\psi(x) \rightarrow \tilde{\psi}(x) := e^{i\alpha(x)} \psi(x)$$

$$A_\mu(x) \rightarrow \tilde{A}_\mu(x) := A_\mu(x) - i \partial_\mu \alpha(x)$$

the action obeys:

$$S'[\psi, A] \rightarrow S'[\tilde{\psi}, \tilde{A}]$$

$$= \int \bar{\psi}(x) e^{-i\alpha(x)} \left(i \gamma^\mu (\partial_\mu + i A_\mu - i \partial_\mu \alpha - m) \right) e^{i\alpha(x)} \psi(x) d^4x$$
$$= S'[\psi, A]$$

Generalization to Yang-Mills theory

Gauging $\psi(x) \rightarrow e^{i\alpha(x)} \psi(x)$ introduced $A_\mu(x)$.

and $A_\mu(x)$ turns out to exist: The EM 4-potential.

We "derived" the electromagnetic force!

Notice: $e^{i\alpha(x)} \in U(1)$

$$U(1) = \{ G \in \mathbb{C} \mid G^\dagger = G^{-1} \}$$

Now give the Dirac particles an extra index (isospin bundle)

$$S'[\psi] = \int \bar{\psi}_a \left(i g \gamma^\mu \delta_{ab} \partial_\mu - m \delta_{ab} \right) \psi_b d^4x \quad \left(\sum_{a,b} \text{implied} \right)$$

It's invariant under:

$$\psi_a(x) \rightarrow G_{ab} \psi_b(x) \quad \left(\sum_{b=1}^N \text{implied} \right)$$

where $G \in SU(N)$

$$SU(N) = \{ G \in M_n(\mathbb{C}) \mid G^\dagger = G^{-1}, \det(G) = 1 \}$$

Now, we gauge, i.e., require invariance under:

$$\psi_a(x) \rightarrow G_{ab}(x) \psi_b(x) \quad \text{where } G \in SU(N)$$

$\Rightarrow$ Invariance of the action now requires new field $B_\mu(x)$:

$$S'[\psi] = \int \bar{\psi}_a \left(i g \gamma^\mu \underbrace{\left(\delta_{ab} \partial_\mu + i B_\mu(x) T_{ab} \right)}_{\text{"covariant derivative"}} - m \delta_{ab} \right) \psi_b d^4x$$

and $B_\mu(x)_r \rightarrow \tilde{B}_\mu(x)_r = B_\mu(x)_r + \text{complicated}$

Here: $T_{ab} \in \mathfrak{su}(N)$ are a Lie algebra basis, i.e. they are generators of infinitesimal $SU(N)$ transformations.

Upshot: $\square$ $N=2$ Weak force (though mixed with $N=1$ EM)
 $\square$ $N=3$ Strong force QCD.

Recall:

$$\int_{\text{man}} (\theta^r) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu} \quad \text{Einstein action}$$

$$-\frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^{\mu\nu\sigma} = 8\pi G *T_\mu \quad \text{Einstein equation}$$

are the same also with any choice of ON bases in the tangent spaces, i.e., we have a local symmetry under:

$$\theta^\mu(x) \rightarrow \tilde{\theta}^\mu(x) = A^\mu{}_\nu(x) \theta^\nu(x)$$

The $A^\mu{}_\nu(x)$ are **local Lorentz transformations**.

Our covariant derivative:

$$\nabla_{e_\mu} (v^\nu(x) e_\nu) = \left(\frac{\partial}{\partial x^\mu} v^\nu(x) \right) e_\nu + v^\nu(x) \underbrace{\omega^\rho{}_\nu(e_\mu)}_{\text{plays role of } A_\mu, B_\mu} e_\rho$$

Do the $\omega^\rho{}_\nu$ indeed generate infinitesimal Lorentz transformations?

Plays rôle of A_μ, B_μ but is now gravity!

→ Interpretation of the connection in ON frames:

Q: The connection 1-forms $\omega^\mu{}_\nu$ are not, we know, tensor-valued 1-forms. Wherin do they take their values?

A: The connection 1-forms take values in the set of infinitesimal Lorentz transformations

Intuition?

The connection yields the change under infinitesimal parallel transport - and parallel transport preserves the metric, i.e. it preserves the lengths of vectors, i.e. the change can only be an infinitesimal "rotation", i.e. an infinitesimal Lorentz transformation.

Recall:

"Lorentz transformations Λ_a^μ " are lin. maps obeying:

$$\Lambda_a^\mu \Lambda_b^\nu \eta_{\mu\nu} = \eta_{ab}$$

$\Rightarrow$ Infinitesimal Lorentz transformations

$$\Lambda_a^\mu = \delta_a^\mu + \varepsilon_a^\mu \quad \text{with } (\varepsilon_a^\mu)^2 = 0$$

why:

$$(\delta_a^\mu + \varepsilon_a^\mu)(\delta_b^\nu + \varepsilon_b^\nu) \eta_{\mu\nu} = \eta_{ab}$$

$$\text{i.e.: } \varepsilon_a^\mu \eta_{\mu b} + \varepsilon_b^\nu \eta_{a\nu} = 0$$

$\Rightarrow$ Infinitesimal Lorentz transformations "JLT" are given by

all $\Lambda_a^\mu = \delta_a^\mu + \varepsilon_a^\mu$ which obey:

$$\varepsilon_{ba} + \varepsilon_{ab} = 0$$

Q: Are connection 1-forms JLT-valued?

Proposition:

In orthonormal frames, the 1-form $\omega_{\mu\nu}$ obeys

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

□ Recall: Absolute exterior derivative: (an anti-derivation)

$$Dt^{a\dots b}_{c\dots d} = dt^{a\dots b}_{c\dots d} + \omega^a_i t^{i\dots b}_{c\dots d} + \dots - \omega^i_c t^{a\dots b}_{i\dots d} - \dots$$

↑
any tensor-valued
differential form.

↑
play the rôle of the Γ^a_{bc}

Thus:

(0,2) tensor-valued 0-form

= 0 because $g_{\mu\nu} = \eta_{\mu\nu} = \text{const}$

can drop the $\wedge$ because g is a 0-form.

$$0 = \nabla g_{\mu\nu} = Dg_{\mu\nu} = dg_{\mu\nu} - \omega^i_\mu \wedge g_{i\nu} - \omega^i_\nu \wedge g_{\mu i}$$

$$\text{i.e. } 0 = \omega_{\nu\mu} + \omega_{\mu\nu} \quad \checkmark$$

Recall that by using a tetrad, we achieved that $g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1_i \end{pmatrix} = \eta_{\mu\nu}$ everywhere!