

Classification of solutions of GR

(and along the way we will introduce some generally useful methods of group theory)

Recall:

- The task is to solve the equations of motion of matter, jointly with the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- In practice, this problem must be simplified, i.e., the number of to-be-determined functions must be reduced.

⇒ Make symmetry assumptions.

Question: How much can we weaken the symmetry assumptions of Friedmann-Lemaître and still get exact solutions?

Strategy:

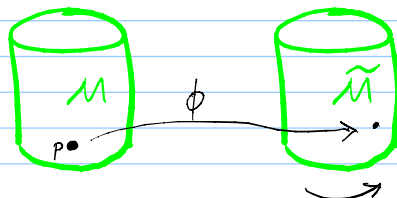
- Classify cosmological models (M, g) , $T_{\mu\nu}$ by the amount and type of symmetry assumed.
- For each amount and type of symmetry assumed, try to find exact solutions or at least (asymptotic) properties of exact solutions.

Remark: Among the high symmetry models, some come arbitrarily close to F.L. at finite times!

See, e.g., text by Wainwright & Ellis.

Recall: Symmetries & Killing vector fields

- Two spacetimes (M, g) , $(\tilde{M}, \tilde{g})$ are isometric (and therefore of exactly identical shape) if there is a diffeomorphism $\phi: M \rightarrow \tilde{M}$ so that the image of the metric g in $\tilde{M}$ is $\tilde{g}$: $Tg = \tilde{g}$.
- A space-time has a symmetry, if we find such a ϕ for $\tilde{M} = M$.
- Example:



ϕ performs a rotation of M about a symmetry axis, to obtain $\tilde{M} = M$ with $Tg = \tilde{g}$.

Note: The set of all symmetries of a manifold (M, g) forms a "group":

Definition: A "group" G is a set, with an operation, say " $\circ$ ",
 $\circ: G \times G \rightarrow G$

and a "neutral element", say " e ", $e \in G$, such that

$$(a \circ b) \circ c = a \circ (b \circ c) \quad \forall a, b, c \in G$$

$$a \circ e = e \circ a = a \quad \forall a \in G$$

$$\exists a^{-1}: a^{-1} \circ a = a \circ a^{-1} = e \quad \forall a \in G$$

$\uparrow$ "there exists"

$\uparrow$ "for all"

Definition: A group G is called a Lie group if G is also a finite-dimensional smooth manifold.

Example: The sets of rotations in $\mathbb{R}^3$ forms a 3-dimensional Lie group, $SO(3)$.

The angles



α, β, γ are coordinates for elements $g \in SO(3)$.

Remarks: $\square$ The symmetries of a manifold (M, g) can be discrete, such as reflections.

$\square$ But often, the symmetry group of a manifold (M, g) is actually a Lie group.

Note: $\square$ Each $h \in G$ yields an isometric diffeomorphism, by assumption.

$$h: M \rightarrow M, \text{ namely } h: p \rightarrow h(p) \quad \forall p \in M$$

$\square$ Consider the set $\mathcal{O}_p \subset M$ defined by: $\mathcal{O}_p := \{q \in M \mid \exists h \in G: h(p) = q\}$

Definition: The set $\mathcal{O}_p$ is called the *Orbit* of p under the action of the group G .

Note: If G is a Lie group then each orbit $\mathcal{O}_p$ is p or a submanifold of (M, g) .

Question: What are the *infinitesimal* isometric diffeomorphisms?

And what type of mathematical structure do the infinitesimal symmetries form?

$\square$ Recall: The Lie derivative,

$$\begin{aligned} L_{\xi} Q^{a \dots b}_{c \dots d} &= Q^{a \dots b}_{c \dots d; j} \xi^j \\ &\quad - Q^{k \dots b}_{c \dots d} \xi^a_{; k} - \dots - Q^{a \dots k}_{c \dots d} \xi^b_{; k} \\ &\quad + Q^{a \dots b}_{k \dots d} \xi^k_{; c} + \dots + Q^{a \dots b}_{c \dots k} \xi^k_{; d} \end{aligned}$$

yields the rate of change of a tensor Q along the flow of diffeomorphisms ϕ generated by a vector field ξ .

$\Rightarrow$ Here, can use L_{ξ} to differentiate along symmetry group orbits.

$\square$ Thus, if $L_{\xi} g_{\mu\nu} = 0$

then ξ generates isometries $\phi: M \rightarrow M, g \rightarrow \tilde{g} = g$.

$\square$ But $L_{\xi} g_{\mu\nu} = 0 = \xi^{\kappa} g_{\mu\nu;\kappa} + g_{\kappa\nu} \xi^{\kappa}_{;\mu} + g_{\mu\kappa} \xi^{\kappa}_{;\nu}$
 ξ^{κ} always for Γ, g compatibility
 i.e. is itself an infinitesimal symmetry

$\Rightarrow$ A vector field ξ generates a symmetry of spacetime if it is a Killing vector field:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \quad (X)$$

Q: Maximum number, d , of Killing vector fields in n dims.?

A: $d = n(n+1)/2$ To see this, note that there are 2 ways to obey Eq. (X):

a) $\xi_{\mu;\nu} = 0 \quad \forall \nu$, i.e. $\nabla \xi = 0$

(can have maximally n such indep. vectors)

b) $\nabla \xi \neq 0$, but then $K_{\mu\nu} := \xi_{\mu;\nu}$ is antisymmetric

(can have at most $n(n-1)/2$ indep. such cases.)

$\Rightarrow d = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$

From a symmetry Lie group to a "symmetry Lie algebra":

General idea:

Normally the points of a manifold cannot be multiplied!

$\square$ A Lie group is a smooth manifold with extra structure: the multiplication.

$\square$ Notion: Product of group elements close to $1 \in G$ yields a group element close to 1 .

$\square$ Consider the tangent space $T_1(G)$ to the point $1 \in G$ of the Lie group manifold G .

$\square$ $T_1(G)$ is a vector space and it has extra structure, inherited from the group's multiplication.

$\square$ Define the Lie algebra of a group G to be $T_1(G)$, equipped with the inherited "multiplication".

The identity element of the group, $p=1$ is also a point of the group's manifold. $T_1(G)$ is the tangent space to this point.

Crucial fact: From knowledge of only the Lie algebra, i.e., only $T_1(G)$ and its "multiplication", the group G can be constructed!
(though not always uniquely)

- Let us collect the properties that the inherited multiplications of all Lie algebras share.
- Then, let us define **Lie algebras** as anything with these properties:

Definition:

A Lie algebra is a vector space A , with an operation $\{, \}$

$$\{, \} : A \times A \rightarrow A \quad \text{"Lie bracket"}$$

obeying $\{r, s\} = -\{s, r\} \quad \forall r, s \in A$

and $\{\{r, s\}, t\} + \{\{t, r\}, s\} + \{\{s, t\}, r\} = 0$ "Jacobi identity"

Theorem: Every vector space A with a "multiplication" $\{, \}$ that obeys these axioms is isomorphic to $T_2(G)$ of a Lie group G .

Proposition: The set of Killing vector fields $\xi^{(i)}$ of (M, g) is a Lie algebra.

Exercise: Prove this, i. e., show the following:

Assume $\xi^{(1)}, \xi^{(2)}$ are Killing vector fields of (M, g) and $\alpha, \beta \in \mathbb{R}$.

Then: $\alpha \xi^{(1)} + \beta \xi^{(2)}$ (i.e., they form a vector space)

and $\{\xi^{(1)}, \xi^{(2)}\} := \xi^{(1)}\xi^{(2)} - \xi^{(2)}\xi^{(1)}$

are also Killing vector fields,

and the $\xi^{(i)}$ obey the Jacobi identity.

Summary of the big picture:

1. The symmetries of any (M, g) form a group: they can be concatenated associatively, and all possess an inverse. ↓ Recall: there can be discrete symmetries too. Some symmetries are differentiable, parametrized by the flow $\Rightarrow$ the symmetries form a Lie group.
2. Each Killing vector field is the infinitesimal generator of a flow of isometric diffeomorphisms, i.e., of a symmetry.
3. We see here that the Killing vector fields indeed form a Lie algebra.
4. Recall that every Lie algebra generates a Lie group.

Surfaces of homogeneity and the isotropy subgroup:

□ Definition:

Let r be the dimension of the Lie algebra, i.e., also the dimension of the Lie group of symmetries.

□ Recall this definition:

- Consider the set of points $\mathcal{O}(p)$ that a point p can flow to along the Killing vector fields.
- $\mathcal{O}(p)$ is called the orbit of $p \in M$ under the action of the symmetry group. We denote the dimension of the orbit by s .

Clearly:

The dimension of an orbit cannot be larger than the dimension of the symmetry group, i. e.

$$s \leq r,$$

but $s < r$ easily happens:

Example:

Consider $M := \mathbb{R}^2$ and $p = (0,0)$.

Then $r = r_{\max} = \overset{n=2}{n(n+1)/2} = \underline{\underline{3}}$ is dim. of sym. group.

$\Rightarrow$ The three-dimensional Lie algebra of Killing vector fields is spanned by three Killing vector fields:

Concretely:

$$K^{(1)} := \frac{\partial}{\partial x}, \quad K^{(2)} := \frac{\partial}{\partial y}$$

$$K^{(3)} := y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$$

Orbit of $p = (0,0)$:

(Group elements generated by them are $e^{a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}}$ and they act as $e^{a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}} f(x,y) = f(x+a, y+b)$ by Taylor expansion.)

$O(p) = \mathbb{R}^2$ because generators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ generate flow to every where.

Def: The surface of homogeneity has dimension $s = 2 < r$
 $\uparrow$ generated by the Killing vectors (here: $K^{(1)}, K^{(2)}$) which do not have trivial orbits

Notice: Since $n=2$, at any given point p , only at most 2 Killing vectors can be linearly independent at p .

□ Role of $K^{(3)}$?

($K^{(3)}$ is the angular momentum
and it of course generates rotations:
 $e^{2K^{(3)}} f(x,y) = f(x \cos 2 - y \sin 2, x \sin 2 + y \cos 2)$)

The flow generated by $K^{(3)}$ leaves p fixed
and rotates everything around p .

□ Definition:

We say that those Killing vector fields
which do not generate a homogeneity surface,
i.e., which generate a trivial group orbit for a point
are generating the isotropy subgroup (of the
full symmetry group generated by all Killing vectors).

□ Dimension, d , of the isotropy subgroup?

Clearly: $d = r - s$
 $\uparrow$ isotropy $\uparrow$ full $\uparrow$ homogeneous.

Classification of cosmological models

□ The classification is with respect to:

□ Dimension of isotropy subgroup d :

(# of conserved 'angular momenta') $\rightarrow$ $d = 0, 1, 2, 3, 4, 5, 6$
 $\swarrow$ anisotropic case $\uparrow$ at each $p \in M$ one rotational symmetry axis $\nwarrow$ e.g. full Lorentz symmetry
e.g. spatially isotropic case

□ Dimension of homogeneity surfaces s :

(# of conserved 'momenta') $\rightarrow$ $s = 0, 1, 2, 3, 4$
 $\swarrow$ inhomogeneous $\uparrow$ homogeneous on 3dim orbits $\nwarrow$ homogeneous

□ A large body of literature exists on most cases of (d, s) :

□ Many exact solutions are known!

□ Many asymptotic behaviors are known!

□ Comprehensive text:

Wainwright & Ellis, *Dyn. systems in cosmology*,
Cambridge Univ. Press (1997)

□ Examples:

	homogeneity ↓	isotropy ↓	
□	<u>5</u>	<u>d</u>	
	4	3	Einstein's static model
	4	1	Gödel's model
	4	0	Ostrovskii-Ken models
	3	3	Friedmann-Lemaître models
	3	1	spatially hom & locally one rot. sym axis
	3	0	Bianchi models
	⋮	⋮	

Powerful alternative classification approach:

Idea: Classify the possible $T_{\mu\nu}$, then use Einstein equation to obtain classification of curvature.

Proposition:

For every physical energy momentum tensor $T_{\mu\nu}$ there exists a unique timelike vector field u so that $T_{\mu\nu}$ takes this standard form:

$$T_{ab} = \overset{\text{scalar}}{\mu} u_a u_b + \overset{\text{vector}}{q_a} u_b + \overset{\text{vector}}{q_b} u_a + \overset{\text{scalar}}{p} (g_{ab} + u_a u_b) + \overset{\text{tensor}}{\pi_{ab}}$$

where q and π are a vector field and a tensor field obeying:

$$q_a u^a = 0, \quad \pi_{ab} u^b = 0, \quad \pi_a^a = 0, \quad \pi_{a,b} = \pi_{b,a}$$

Definition: u is called the "fundamental 4-velocity field"

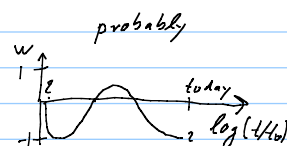
Note: E.g., for a perfect fluid this is the fluid velocity:

$$T_{ab} = \rho u_a u_b + p(g_{ab} + u_a u_b), \quad u_a u^a = -1$$

Recall: equation of state is

$$p = (\gamma - 1)\rho$$

$$\gamma = \begin{cases} 1 & \text{dust} \\ 4/3 & \text{radiation} \\ 0 & \text{cosmological constant} \end{cases}$$



□ Definition:

If (M, g) possesses spacelike $s=3$ homogeneity but the fundamental velocity is not orthogonal to the homogeneity surfaces, then we say that this cosmology is "tilted".

Segré classification:

- A systematic classification of $T_{\mu\nu}$ can be performed, by the analysis of its eigenvalues / eigenvectors. *Nontrivial because:*
- $T_{\mu\nu}$ is symmetric.
But, the inner product in the vector space is $g_{\mu\nu} \Rightarrow T_{\mu\nu}$ is generally not hermitian!
- $T^\mu{}_\nu$ is in a space with the inner product $g^\mu{}_\nu = \delta^\mu{}_\nu$, but $T^\mu{}_\nu$ is generally not symmetric!

Use Jordan normal form:

$\Rightarrow$ Segré classification yields 4 main types of energy momentum tensors $T_{\mu\nu}$.

Recall strategy:

The classification of possible $T_{\mu\nu}$ should, via the Einstein eqns, yield a classification of possible curvatures.

Indeed: In $3+1$ dimensions the Einstein equation also reads:

$$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)$$

Exercise: Prove this and notice the dimension-dependence

$\Rightarrow$ The 10 degrees of freedom of $T_{\mu\nu}$ (as a symmetric 4×4 matrix) determine the 10 degrees of freedom of $R_{\mu\nu}$.

$\Rightarrow$ The Segre classification of possible $T_{\mu\nu}$ yields, via the Einstein equation also a classification of possible Ricci tensors $R_{\mu\nu}$.

Q: Does this yield also a classification of the possible Riemann tensors $R^{\mu}{}_{\nu\alpha\beta}$?

A: No! The Ricci tensor contains only 10 of the 20 degrees of freedom of the Riemann tensor! (In $3+1$ dim)

Prop.: The information in $R^{\mu}{}_{\nu\alpha\beta}$ is shared among the Ricci tensor $R_{\mu\nu}$ and the so-called Weyl tensor, $C^{\mu}{}_{\nu\alpha\beta}$.

$\Rightarrow$ It remains to classify the possible Weyl tensors!

The Weyl tensor, C^{am}_{sq} :

$$C^{am}_{sq} := R^{am}_{sq} - \frac{1}{2} (g^s R^m_q + g^m R^s_q - g^m R^s_q - g^s R^m_q) + \frac{1}{6} (g^s g^m - g^s g^s) R$$

Notice: If R^a_b and C^{am}_{sq} are given, they determine R^{am}_{sq} fully:

$$R^{am}_{sq} = C^{am}_{sq} + \frac{1}{2} (g^s R^m_q + g^m R^s_q - g^m R^s_q - g^s R^m_q) - \frac{1}{6} (g^s g^m - g^s g^s) R$$

→ R^{am}_{sq} is expressed through C^{am}_{sq} and R^a_b
 ↑ 20 indep. components ↑ 10 indep. comp. ↑ 10 indep. comp.

⇒ The Weyl tensor C^{am}_{sq} indeed contains all that information about the curvature R^{am}_{sq} which is not in R^a_b .
 Determined from $T_{\mu\nu}$ via the Einstein eqn.

⇒ C^{am}_{sq} contains all that curvature information which is not determined via the Einstein equation by $T_{\mu\nu}$.

⇒ C^{am}_{sq} describes all that curvature which can exist even where there is no matter! (e.g.: gravity waves)

Proposition

□ Assume (M, g) is a 3+1 dimensional Lorentzian manifold.

□ Choose any smooth positive scalar function ϕ on M .

□ Define $(M, \tilde{g})$ with the new metric $\tilde{g}$ obtained through the "conformal transformation":

$$g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(x) := \phi(x) g_{\mu\nu}(x)$$

Intuition:

Weyl curvature distorts (000)

but only Ricci curvature

shrinks or expands overall: (000)

Then: $\tilde{C}^{\alpha\beta\gamma\delta}(x) = C^{\alpha\beta\gamma\delta}(x) \quad \forall x \in M$ (Exercise: what would be a proof strategy?)

Historical remark

- Consider the equivalence class of spacetimes $(M, \tilde{g})$ that are conformally equivalent to Minkowski space:

$$g_{\mu\nu}(x) = \phi^2(x) \eta_{\mu\nu}$$

- Einstein and Fokker initially considered a theory in which the metric possesses only this conformal degree of freedom ϕ (to play role of Newton's gravitational potential).

Newton gravity does come out correctly as a limiting case!

- Then, $S = \int_M R \sqrt{g} d^4x + \int_M \mathcal{L}_{\text{matter}} \sqrt{g} d^4x$ and $\frac{\delta S}{\delta \phi} = 0$ yield:

$$R = 8\pi G T^\mu{}_\mu$$

↳ In electromagnetism $T^{(EM)\mu}{}_\mu = 0$
i.e. EM fields would not gravitate.

No gravity waves here because $C_{cd}^{ab} = C_{cd}^{ab(\text{Minkowski})} = 0$

- Equivalence principle ok.

- Light bending & Mercury perihelion shift wrong.

Recall: via the Einstein equation the Segré classification implies a classification of properties of the Ricci tensor $R_{\mu\nu}$.

It remains to classify the Weyl tensor:

Petrov classification:

This is a classification of the Weyl tensor $C^{\mu\nu}{}_{\rho\sigma}$, which possesses the 10 remaining degrees of freedom of $R^{\mu\nu}{}_{\rho\sigma}$.

- $C^{\mu\nu}{}_{\rho\sigma}$, just like the Riemann tensor, is antisymmetric in $\mu \leftrightarrow \nu$ and in $\rho \leftrightarrow \sigma$, and symmetric in $\mu\nu \leftrightarrow \rho\sigma$.

Thus $C^{\mu\nu}_{\sigma\epsilon}$ can locally be viewed as a symmetric map from the antisymmetric part $A_p(M)^2$ of $T_p(M)^2$ (so called bi-vectors) into itself:

$$C: A_p(M)^2 \rightarrow A_p(M)^2$$

But, the inner product in $A_p(M)^2$ is not positive definite!

$\Rightarrow C$ is generally not hermitean.

Therefore, use Jordan normal form again:

Result: 6 main Petrov classes for Weyl curvature:
according to eigenvalue/eigenvector decomposition.

Type O: Weyl curvature vanishes

Type D: "Static" Weyl curvature, e.g., in vicinity of a star.

Type N: Transverse gravitational waves, the type LIGO aims to detect. Like light, their strength decays $\sim \frac{1}{r}$ from the source.

Type I: Longitudinal gravitational waves

These waves cause a shear effect.

However, they decay fast: $\sim \frac{1}{r^2}$

Why? Gravitational waves, when small enough, travel with speed of light. Like light, they then cannot oscillate longitudinally.

Types II, III: Mixtures of the above.

□ Potential problem: (with symmetry assumptions):

(E.g. recall that flatness
in FL spacetimes is unstable)

□ The so-obtained highly symmetric solutions, e.g. Friedman-Lemaître, may possess properties that are peculiar to high symmetry.

(E.g.:
In Newtonian gravity, a slightly
non-symmetric collapse of a star
would not lead to a singularity
but to a bounce - think figure skater.)

□ E.g.: When a Friedmann-Lemaître solution, or a Schwarzschild solution exhibits a singularity: Is it due to symmetry, or realistic?

□ Singularity theorems (see later) confirm the robustness under certain conditions (such as strong energy condition).

→ More confidence in significance of the properties of highly symmetric solutions.