

Recall:

Given, in particular, the strong energy condition, our singularity theorem claimed that geodesics meet a divergence of a quantity called **expansion**, θ , in finite proper time in the past and this will mean a big bang singularity:

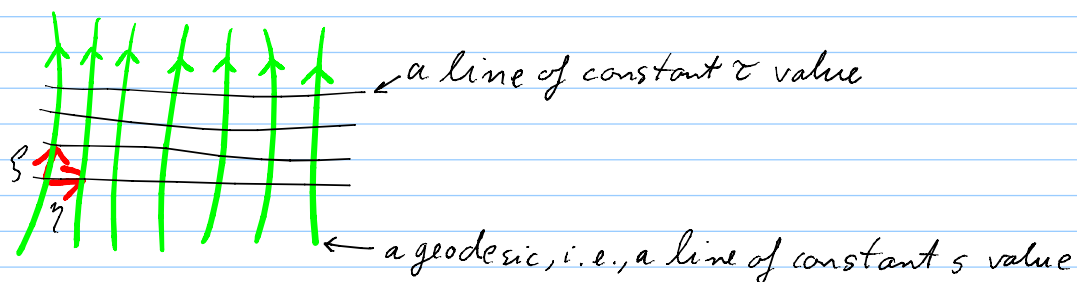
The "expansion", θ : important notion also e.g. in study of grav. collapse of stars.

- Consider a "congruence of timelike geodesics" e.g., freely falling dust. through Σ , i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$: (Σ is a (cauchy) surface)

- We consider a one-parameter sub-family of these geodesics:

$$x(\tau, s)$$

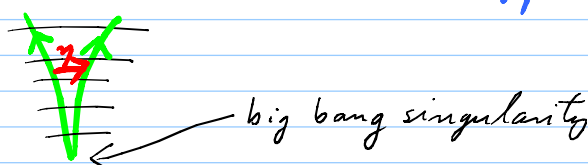
$\uparrow$ parameter of family of neighboring geodesics.
 eigentime



- Then, we define the deviation vector to a neighboring geodesic:

$$\eta := \frac{dx}{ds}$$

- The singularity theorem claims that this happened in the past:



How does η change along a past-directed timelike geodesic with tangent ξ ?

We showed:

$$\xi^\mu \eta^\nu{}_{;\mu} = \eta^\mu B^\nu{}_\mu \quad \text{where} \quad B^\nu{}_\mu := \xi^\nu{}_{;\mu}$$

$\Rightarrow$ Along the geodesic, ξ , the deviation vector η^μ changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

□ The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely:

$$B_{\mu\nu} = \underbrace{\omega_{\mu\nu}}_{\text{antisymmetric}} + \underbrace{\sigma_{\mu\nu}}_{\text{Symmetric and trace}=0} + \underbrace{\epsilon_{\mu\nu}}_{\text{rest}}$$

Explicitly:

Volume preserving $\rightarrow$	$\omega_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} - B_{\nu\mu})$	Twist: $\circ \rightarrow \circ$
	$\sigma_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \theta h_{\mu\nu}$	Shear: $\circ \rightarrow \text{!}$
Volume changing:	$\epsilon_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu}$	Expansion: $\circ \rightarrow \bigcirc$

Here, we defined: $\theta := B^{\mu\nu} g_{\mu\nu}$ and $h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$

I.e., the Expansion, θ , is the trace of B , which we showed is also equal to the magnitude of the spatial part of B : $\theta = B^{\mu\nu} h_{\mu\nu}$.

Key question:

What is the dynamics of θ ?

The Raychaudhuri equation

For the derivation, we will use:

A) Definition of B is: $B_{\mu\nu} := \xi_{\mu;\nu}$

B) The curvature tensor obeys the Ricci equation:

$$\xi^a{}_{jbc} - \xi^a{}_{jcb} = R^a{}_{bcd} \xi^d$$

C) ξ is tangent to a geodesic, i.e., it obeys: $\nabla_{\xi} \xi = 0$

$$\text{i.e.: } 0 = \nabla_{\xi e_a} \xi^b e_b = \xi^a \nabla_{e_a} \xi^b e_b = \xi^a \xi^b{}_{;a} e_b$$

$$\text{True for all } e_a, \text{ thus: } \xi^a \xi^b{}_{;a} = 0$$

Now calculate the rate of change of B along the geodesic:

$$\begin{aligned} \xi^c B_{ab;c} &\stackrel{(A)}{=} \xi^c \xi_{a;jc} \\ \nabla_{\xi} B &\stackrel{(B)}{=} \xi^c \xi_{a;jc} + \xi^c R_{abcd} \xi^d \end{aligned}$$

$$\stackrel{\text{Leibniz rule}}{=} \underbrace{(\xi^c \xi_{a;jc})_{;b}}_0 - \xi^c{}_{;b} \xi_{a;c} + R_{abcd} \xi^c \xi^d$$

$$\stackrel{(C)}{=} -\xi^c{}_{;b} \xi_{a;c} + R_{abcd} \xi^c \xi^d$$

$$\stackrel{(A)}{=} -B^c{}_b B_{ac} + R_{abcd} \xi^c \xi^d$$

In summary, we derived:

$$\xi^c B_{ab;c} = -B^c_b B_{ac} + R_{abcd} \xi^c \xi^d \quad (*)$$

The trace of (*) will be the Raychaudhuri equation.

But first, we recall:

$$\square \quad \xi = \frac{d}{d\tau}$$

$$\square \quad \text{Tr } B = B_{\mu\nu} g^{\mu\nu} = \Theta$$

$$\Rightarrow \text{Trace(LHS) of } (*) \text{ reads } \frac{d}{d\tau} \Theta !$$

Now on the RHS of (*) use the decomposition

$$B_{\mu\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3} \Theta h_{\mu\nu} \text{ to express } B^c_b B_{ac}:$$

$$\begin{aligned} B^c_b B_{ac} &= \omega^c_b (\omega_{ac} + \sigma_{ac} + \frac{1}{3} \Theta h_{ac}) \\ &+ \sigma^c_b (\omega_{ac} + \sigma_{ac} + \frac{1}{3} \Theta h_{ac}) \\ &+ \frac{1}{3} \Theta h^c_b (\omega_{ac} + \sigma_{ac} + \frac{1}{3} \Theta h_{ac}) \end{aligned}$$

When taking the trace, $g^{ab} B^c_b B_{ac}$, only the diagonal terms survive:

$$\text{Tr}(BB) = g^{ab} B^c_b B_{ac} = \omega_{ab} \omega^{ab} + \sigma_{ab} \sigma^{ab} + \frac{1}{3} \Theta^2 h_{ab} h^{ab}$$

Exercise: show it is 3

The Raychaudhuri equation is then the trace of Eq. (*):

$$\frac{d\Theta}{d\tau} = -\frac{1}{3} \Theta^2 - \underbrace{\sigma_{ab} \sigma^{ab}}_{\text{always positive}} - \underbrace{\omega_{ab} \omega^{ab}}_{\text{always positive (and vanishes if choose congruence } \perp \Sigma)} - \underbrace{R_{cd} \xi^c \xi^d}_{\text{pos. or neg?}}$$

recall: Ricci tensor is $R_{cd} = R_{da}$

↪ ?

Dynamics

□ Assume that

$$R_{\mu\nu} \xi^\mu \xi^\nu \geq 0 \quad \text{for all timelike } \xi$$

i.e., using the Einstein equation

$$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^a_a)$$

we are assuming that

$$T_{\mu\nu} \xi^\mu \xi^\nu - \frac{1}{2} \xi^\mu \xi_\mu T \geq 0 \quad \text{whenever } \xi^\mu \xi_\mu < 0$$

i.e. the Strong Energy Condition.

Thus, assuming the strong energy condition:

$$\frac{d\theta}{d\tau} + \frac{1}{3} \theta^2 \leq 0$$

$$\text{i.e., } -\frac{1}{\theta^2} \frac{d\theta}{d\tau} - \frac{1}{3} \geq 0$$

$$\text{i.e., } \boxed{\frac{d}{d\tau} \theta^{-1} \geq \frac{1}{3}} \quad (+)$$

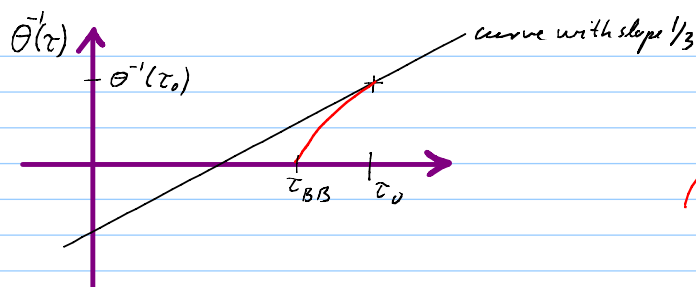
Consider the cases when the geodesics are initially all either

a.) diverging, i.e., $\theta(\tau_0) > 0$ (expanding universe) or

b.) converging, i.e., $\theta(\tau_0) < 0$ (contracting universe)

(This is reformulating the theorem's assumption that the extrinsic curvature (i.e. the expansion or contraction at some time exceeds a certain finite value everywhere)

a.)

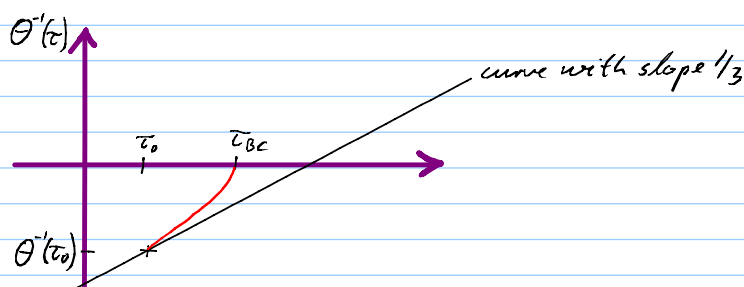


$\tau_0 = \text{e.g. today}$

red curve = curve $\theta'(\tau)$ of slope $> 1/3$

We see that $\theta'(\tau)$ must have hit $\theta'(\tau)=0$ at a finite time τ_{BB} (Big Bang).

b.)



$\tau_0 = \text{e.g. today}$

red curve = curve of slope $> 1/3$

We see that $\theta'(\tau)$ will hit $\theta'(\tau)=0$ at a finite time τ_{BC} (Big Crunch)

Conclusion: Eq. (+) implies that $\theta(\tau)$ must go through 0, i.e.:

a.) for sufficiently early τ , have $\theta \rightarrow +\infty$, i.e.: Big Bang

b.) for sufficiently late τ , have $\theta \rightarrow -\infty$, i.e.: Big Crunch

Note:

This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.

Remark:

Singularity theorems suitable for black hole case also assume a trapped surface, i.e., a surface on which both in- and outgoing null geodesics converge.

Results, e.g., regarding cosmic singularity?

- ▣ Assume a set of symmetries of matter and spacetime has been chosen.
 - ▣ Assume an exact solution or at least its asymptotic properties at early times have been found.
 - ▣ Assume, we choose a timelike congruence e.g. of geodesics (or of the fundamental velocity field - they may be different)
- ⇒ We can now explicitly calculate the **twist**, **shear** and **expansion** along the congruence:

The Hubble functions:

In particular, we can see how the expansion or contraction of the universe behaves dynamically, e.g. when the condition of perfect isotropy is relaxed:

- ▣ Now we have different expansions in different directions, nonlinearly influencing another.
- ▣ Recall:
The expansion in one direction can be say enhanced by shear, as long as shear shrinks other directions.
- ▣ Definition:
We define a rate of expansion tensor that includes shear:

$$\theta_{\mu\nu} := \underbrace{\sigma_{\mu\nu}}_{\text{symmetric part of } B_{\mu\nu}} + \underbrace{\frac{1}{3} \theta h_{\mu\nu}}_{\substack{\text{projector } \perp \text{ to the} \\ \text{timelike } u\text{-field} \\ \text{expansion scalar.}}} \quad \text{shear}$$

$\theta_{\mu\nu}$ is fully spacelike and symmetric $\Rightarrow \theta_{\mu\nu}$ can be diagonalized in suitable ON frame $\{e_0, e_1, e_2, e_3\}$:

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \theta_1 & 0 & 0 \\ 0 & 0 & \theta_2 & 0 \\ 0 & 0 & 0 & \theta_3 \end{pmatrix} \quad \text{3 space-like directions.}$$

with the traditional expansion being the trace (because $\sigma_{\mu\nu}$ is traceless):

$$\theta = \theta_1 + \theta_2 + \theta_3 \quad \Rightarrow \text{is not quite projector}$$

why $\frac{1}{3}$? Recall that $\text{Tr}(h_{\mu\nu}) = 3$

Definition:

$$H_i := \frac{1}{3} \theta_i$$

Local Hubble expansion function in direction e_i .

$$H := \frac{1}{3} \theta$$

Overall local Hubble expansion function.

Definition:

We use H_i, H to define local directional and general scale factors l_i, l :

The l_i, l are defined as the solutions to:

$$\frac{\dot{l}_i}{l_i} = H_i$$

$$\frac{\dot{l}}{l} = H$$

Here, the time derivative is defined as:

$$\dot{l} = u(l) = u^\nu \frac{\partial}{\partial x^\nu} l \quad \text{recall: } u \text{ is timelike.}$$

What behavior can occur in the far past?

Full set of cases not yet known.

But:

Explicit examples are known where e.g:

- All $l_i \rightarrow 0$ as in FL cosmologies
- $l_1, l_2 \rightarrow 0, l_3 \rightarrow \infty$ "cigar singularity"
- $l_1, l_2 \rightarrow 0, l_3 \rightarrow \text{const}$ "barrel singularity"
- $l_1, l_2 \rightarrow \text{const}, l_3 \rightarrow 0$ "pancake singularity"

□ Note: For homogeneous, isotropic FL models, H is the regular Hubble parameter and l is its scale factor.

Recall: Singularity theorems suitable for black holes involve the concept of a trapped surface.

Def: Let Σ be a spacelike hypersurface. (Note: Σ is 3-dimensional)

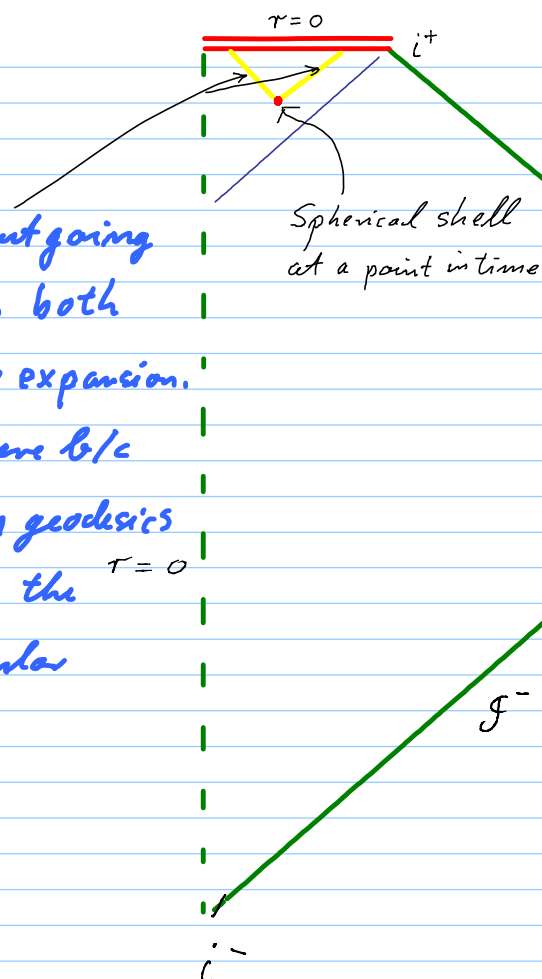
Let $T \subset \Sigma$ be a compact, 2-dimensional smooth spacelike submanifold of Σ . Consider the ingoing and the outgoing future-directed null geodesics that are orthogonal to T .

If all these geodesics possess negative expansion, $\theta < 0$, then T is called a trapped surface.

Examples:

All concentric spheres inside a Schwarzschild black hole.

The in- and outgoing null geodesics both have negative expansion. Can't see it here b/c the neighboring geodesics are neighbors in the suppressed angular directions.



Note: For $r < 2m$, r is timelike and t is spacelike. Both ingoing and outgoing null geodesics have decreasing r , and $\Theta < 0$.

Def: Let Σ be a spacelike hypersurface.

Then, the (3-dim. spacelike) union, $\mathcal{T}$, of all trapped surfaces $T \subset \Sigma$ is called the **trapped region** of Σ .

Def: The boundary $\partial\mathcal{T} \subset \Sigma$ is called the **apparent horizon** of the spacelike hypersurface Σ .

Note: $\partial\mathcal{T}$ is 2-dimensional and spacelike.

Def: If we foliate spacetime into spacelike hypersurfaces

$$\Sigma_\alpha, \alpha \in I \subset \mathbb{R}$$

each with its apparent horizon, $\mathcal{T}_\alpha$, then their union

$$\mathcal{A} := \bigcup_\alpha \mathcal{T}_\alpha$$

is called the **Trapping horizon** of the spacetime.

Remarks:

- To check for the existence of an event horizon

j^- (worldline to i^+)

in principle requires knowledge of the full future.

- But one can check for the existence of an apparent horizon in any spacelike hypersurface by calculating the expansions only at that time!

- For static Schwarzschild black holes the event and apparent horizons coincide.

- For general black holes, apparent horizons are on or inside the event horizons.