

Recall: Physical motivation for the "Metric Tensor"

□ In Minkowski space, in inertial and cartesian coordinates:

$$\begin{aligned} \overset{\substack{\text{4-dim spa-time} \\ \text{distance!}}}{\left[\text{distance}(x, \hat{x}) \right]^2} &= -(x^0 - \hat{x}^0)^2 + (x^1 - \hat{x}^1)^2 + (x^2 - \hat{x}^2)^2 + (x^3 - \hat{x}^3)^2 \\ &\quad \uparrow \\ &\quad \text{indep. of choice} \\ &\quad \text{of inertial cds.} \\ &= \eta_{\mu\nu} (x^\mu - \hat{x}^\mu) (x^\nu - \hat{x}^\nu) \\ &\quad \text{with } \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

□ In Minkowski space, in an arbitrary coordinate system:

$$\begin{aligned} \left[\text{distance}(x, \hat{x}) \right]^2 &= g_{\mu\nu}(x) (x^\mu - \hat{x}^\mu) (x^\nu - \hat{x}^\nu) + \mathcal{O}^3 \\ &\quad \uparrow \\ &\quad \text{complicated higher} \\ &\quad \text{order terms} \end{aligned}$$

(e.g. polar cds, or accelerated cds) with $g_{\mu\nu}(x) \neq \eta_{\mu\nu}$

□ Generalization to curved space-time, historically:

Allow even such $g_{\mu\nu}(x)$ which in no coordinate system obey:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \text{ for all } x \in \mathcal{M}$$

$\Rightarrow g_{\mu\nu}(x)$ is not simply $\eta_{\mu\nu}$ in noninertial coordinates

$\Rightarrow$ Such $g_{\mu\nu}(x)$ take us beyond special relativity!

□ Enforce Einstein's equivalence principle:

Require $g_{\mu\nu}$ to be such that

for each $x \in \mathcal{M}$ there exists a coordinate system so that at least at x :

(Recall equiv. principles (EP):
If freely falling small masses
fall equally $\Rightarrow$ "weak EP"
+ same internal non-grav. physics $\Rightarrow$ "Einst. EP"
+ same internal grav. physics $\Rightarrow$ "strong EP")

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \left(\begin{array}{l} \text{i.e., locally, special relativity holds} \\ \text{dist}(x, \hat{x})^2 = \eta_{\mu\nu} (x^\mu - \hat{x}^\mu) (x^\nu - \hat{x}^\nu) + \mathcal{O}^3 \\ \text{to lowest nontrivial order.} \end{array} \right)$$

Recall: Math. definition of the metric tensor:

□ g is covariant tensor of rank $(0,2)$

(because η is in special relativity)

e.g. $\theta^\mu(x) = dx^\mu$

□ Thus, if n cotangent vector fields $\theta^\mu(x)$ form bases at each point x , then

g is of the form:

$$g(x) = g_{\mu\nu}(x) \theta^\mu(x) \otimes \theta^\nu(x)$$

↑ recall: $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ and $g_{\mu\nu}$ is invertible (since nondegenerate)

□ $g_{\mu\nu}(x)$ invertible $\Rightarrow$ there exists a tensor g^{-1} of rank $(2,0)$:

$$g^{-1}(x) = g^{\mu\nu}(x) \overset{\text{dual basis}}{e_\mu(x)} \otimes e_\nu(x) \text{ with } g^{\mu\nu}(x) g_{\nu\sigma}(x) = \delta^\mu_\sigma$$

$\Rightarrow$ Modern view of the Einsteinian equivalence principle:

Recall: We asked that for each point $p \in M$ there is a coordinate

system so that, when choosing the bases $\{dx^i\}, \{\frac{\partial}{\partial x^j}\}$

then $g(x) = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$, $g_{\mu\nu}(x) = g(x)(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu})$

obeys: $g_{\mu\nu}(p) = \eta_{\mu\nu}$ (in general only at p)

Modern formulation of the Einsteinian equivalence principle:

Independently of any choice of coordinate system:

There are choices of dual bases $\{\theta^\mu(x)\}, \{e_\nu(x)\}$ of $T_x(M), T_x(M)^*$

so that: $g_{\mu\nu}(x) = g(e_\mu(x), e_\nu(x)) = \eta_{\mu\nu} \quad \underline{\forall x \in M}$

Now, knowing distances through $g_{\mu\nu}$, what else follows?

- Distances yield volumes, namely $g_{\mu\nu}(x)$ induces an $\Omega(x)$.
- g, g^{-1} yield duality of covariance and contravariance.
- g yields "Hodge star" $*$: $\Lambda_p \rightarrow \Lambda_{n-p}$ duality.
- $*$ yields $(,)$ making the Λ_p Hilbert spaces.
for Riemannian manifolds
- g yields co-derivative $\delta: \Lambda_p \rightarrow \Lambda_{p-1}$.
- d, δ yield the Laplacian/d'Alembertian $\Delta: \Lambda_p \rightarrow \Lambda_p$
 $\leadsto$ We can formulate wave equations on M !

Proposition:

Given a notion of distance, i.e., a metric, g , this also induces a volume form Ω .

(i.e., a positive $\Omega \in \Lambda_n(M)$, i.e., that when integrated over any portion of M yields a positive number)

Namely:

□ Assume, as always, that M is oriented.

□ Consider a positive chart.

(i.e. has positive $\det(\text{Jacobian})$ with given atlas)

Then:

□ $\Omega := \sqrt{|g|} \, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$
is a well-defined volume form.

$$:= |\det(g_{ij}(x))|$$

Proof: $\square$ Nonzero for all $p \in M$?
Yes, because g is assumed non-degenerate.

$\square$ Well-defined, i.e., is definition chart-independent?

Yes: To see this, change chart: $x \rightarrow \tilde{x}$

Then: $\tilde{g}_{ij}(\tilde{x}(x)) = g_{rs}(x) \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}$ because covariant

i.e., as matrices:

$$\tilde{g} = \left(\frac{\partial x}{\partial \tilde{x}} \right)^T g \left(\frac{\partial x}{\partial \tilde{x}} \right) \quad \text{now take determinant:}$$

$$\Rightarrow |\tilde{g}| = \left| \frac{\partial x}{\partial \tilde{x}} \right|^2 |g| \quad \text{i.e. } |\tilde{g}|^{1/2} = \left| \frac{\partial x}{\partial \tilde{x}} \right| |g|^{1/2}$$

$$\text{Also: } d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n = \det \left(\frac{\partial \tilde{x}}{\partial x} \right) dx^1 \wedge \dots \wedge dx^n$$

$$\Rightarrow |\tilde{g}|^{1/2} d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n = \underbrace{\left| \frac{\partial \tilde{x}}{\partial x} \right| \left| \frac{\partial x}{\partial \tilde{x}} \right|}_{\prod_{i=1}^n 1} |g|^{1/2} dx^1 \wedge \dots \wedge dx^n \quad \checkmark$$

Notation: (Ω is an n -form. What are its coefficients, as a covariant $(0,n)$ tensor?)

$\square$ Define:

$$\epsilon_{i_1, \dots, i_n} := \begin{cases} +1 & \text{if } (i_1, \dots, i_n) \text{ is even permutation of } (1, 2, \dots, n) \\ -1 & \text{if } (i_1, \dots, i_n) \text{ is odd permutation of } (1, 2, \dots, n) \\ 0 & \text{else} \end{cases}$$

unlike in SRT, $\epsilon_{\dots}$ is not canonical, because

Ω is: $\rightarrow$

$\square$ Then, Ω also reads:

$$\Omega = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n \quad (n\text{-form})$$

$$= \sqrt{|g|} \epsilon_{i_1, \dots, i_n} dx^{i_1} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_n}$$

$$=: \Omega_{i_1, \dots, i_n}$$

$$\Omega = \Omega_{i_1, \dots, i_n} dx^{i_1} \otimes \dots \otimes dx^{i_n} \quad (\text{covariant tensor})$$

$\square$ Ω is called the "canonical", or "(pseudo)Riemannian", or "metric", volume form.

Q: Other use of g ?

A: One needs g to formulate d'Alembertian Δ , or $\square$, for wave equations.

Why? a) $\square$ should be non-directional 2nd derivative, but $d^2 = 0$.

b.) need e.g. $\square: \Lambda^0 \rightarrow \Lambda^0$ for Klein Gordon, i.e., need degree of forms conserved by $\square$.

Strategy:

A) Use g for a covariant $\leftrightarrow$ contravariant tensors relation

B) Define a map "Hodge" $*$: $\Lambda_r \rightarrow \Lambda_{n-r}$

C) Define the "Coderivative": $\delta: \Lambda_r \rightarrow \Lambda_{r-1}$

D) Define "Laplacian/d'Alembertian": $\square := d\delta + \delta d$

Then, e.g., the Klein Gordon equation reads:

$$(\square + m^2) \phi = 0$$

A) Covariant $\leftrightarrow$ contravariant tensors equivalence through g :

$\square$ $g(x)$ can be used as a map: by evaluation of one tensor factor:

$$g(x): T_x(M)' \rightarrow T_x(M),$$

$$g(x): \xi^i(x) e_i(x) \rightarrow g_{\mu\nu}(x) \theta^\mu(x) \otimes \theta^\nu(x) (\xi^i(x) e_i(x))$$

$\theta^\nu(e_i) = \delta_i^\nu$

$$= g_{\mu\nu}(x) \underbrace{\xi^i(x)}_{\in T_x(M)} \underbrace{\theta^\mu(x)}_{\in T_x(M)'} \in T_x(M),$$

$\Rightarrow$ For the coefficient

functions we have: $g: \xi^i(x) \rightarrow \omega_\mu(x) = g_{\mu i}(x) \xi^i(x)$ (relative to bases θ^i, e_j)

$\square$ Conversely, g^{-1} acts as:

$$g^{-1}(x): T_x(M) \rightarrow T_x(M)'$$

$$g^{-1}(x): \omega_\mu(x) \rightarrow \xi^i(x) = g^{\mu i}(x) \omega_\mu(x)$$

$\square$ In this way, g, g^{-1} can lower or raise any

tensor index, e.g.: $g: t^{ij}_\kappa \rightarrow t_{i j \kappa} = g_{is} t^{sj}_\kappa$

and: $g^{-1}: \tau^{ij}_\kappa \rightarrow \tau^{i j \kappa} = g^{\kappa s} \tau^{is}_\kappa$

B) The Hodge * map: $\Lambda_p \rightarrow \Lambda_{n-p}$

Recall:
 $\dim(\Lambda_p) = \binom{n}{p} = \binom{n}{n-p}$
 $= \dim(\Lambda_{n-p})$

- Idea:
- each $v \in \Lambda_p$ is a covariant tensor
 - through g it is equivalent to a contravariant tensor $\tilde{v}$
 - can feed Ω with $\tilde{v}$ to obtain $*v \in \Lambda_{n-p}$.

Concretely:

Consider any $v := \frac{1}{p!} v_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \Lambda_p$

Annotations:
 - anything totally antisymmetric (pointing to the wedge product)
 - one could choose other bases. (pointing to the dx terms)
 - convenient because drops out here (pointing to the $v_{i_1, \dots, i_p}$ coefficients)
 - coefficients as a covariant tensor (pointing to the $v_{i_1, \dots, i_p}$ coefficients)

$$= v_{i_1, \dots, i_p} dx^{i_1} \otimes \dots \otimes dx^{i_p}$$

- Use g^{-1} to define a contravariant image of v :

$$\tilde{v} = \tilde{v}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}}$$

where $\tilde{v}^{i_1, \dots, i_p} := g^{i_1 j_1} g^{i_2 j_2} \dots g^{i_p j_p} v_{j_1, \dots, j_p}$

- Apply Ω on $\tilde{v}$:

$$\Omega(\tilde{v}) = \underbrace{\Omega_{i_1, \dots, i_{n-p}}}_{!!} \tilde{v}^{i_1, \dots, i_p} \overbrace{dx^{i_{p+1}} \otimes \dots \otimes dx^{i_n}}^{n-p \text{ factors}} \in \Lambda_{n-p}$$

!!
 $(*v)_{i_{p+1}, i_{p+2}, \dots, i_n}$

- Define $*v := \Omega(\tilde{v})$, i.e.:

$$*v = (*v)_{i_1, \dots, i_{n-p}} dx^{i_1} \otimes \dots \otimes dx^{i_{n-p}}$$

$$*v = \frac{1}{(n-p)!} (*v)_{i_1, \dots, i_{n-p}} dx^{i_1} \wedge \dots \wedge dx^{i_{n-p}}$$

Proposition:

Assume $v \in \Lambda_p$. Then

$$**v = (-1)^{p(n-p)+s} v$$

E.g. $s=1$ for space-time

What is s ? The "signature" of g is $\text{sgn}(g) = (r, s)$, where in diagonal form: $g = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & r \\ & & & -1 & \\ & & & & \ddots & \\ & & & & & -1 & \end{pmatrix}$

Use $*$ to turn $\Lambda(M)$ into an "Inner Product Space":

Definition:

The Hodge $*$ provides a "scalar" (or also called "inner") product for $\Lambda(M)$:

Exercises:

- 1.) Write (α, β) in coordinates
- 2.) Show that $(,)$ is always positive definite on Λ_0 , i.e., $(\alpha, \alpha) > 0 \forall \alpha \in \Lambda_0, \alpha \neq 0$.

$$(\alpha, \beta) := \int_M \underbrace{\alpha}_{p\text{-form}} \wedge \underbrace{* \beta}_{(n-p)\text{-form}}$$

This definition is extended linearly to forms that are lin. comb. of forms of orb. degree, p .

Notes: $\square$ If g is indefinite, then also $(,)$ is indefinite.

$\square$ If g is positive definite, i.e., if M is Riemannian, then $(,)$ is positive definite and Λ becomes a Hilbert space.

c) $(,)$ yields an adjoint for d , the Co-derivative δ :

Recall: For any operator $A: D_A \subset \mathcal{H} \rightarrow \mathcal{H}$ (with D_A dense, i.e., $\overline{D_A} = \mathcal{H}$), its adjoint A^* is defined to have the domain

$$D_{A^*} := \{ v \in \mathcal{H} \mid \exists w \in \mathcal{H} \forall z \in D_A: \langle v, Az \rangle = \langle w, z \rangle \}$$

and this action: $A^*v := w$. We then have:

Definition: $\langle A^*v, z \rangle = \langle v, Az \rangle \quad \forall z \in D_A, v \in D_{A^*}$

The co-derivative, δ , is the (anti-) adjoint of d with respect to the inner product $(,)$ on $\Lambda(M)$:

$$(\delta \alpha, \beta) := -(\alpha, d\beta) \quad \forall \alpha \in D_\delta, \beta \in D_d$$

C) The Codifferential δ explicitly

Clearly: $\delta : \Lambda^p(M) \rightarrow \Lambda^{p-1}(M)$

Proposition: $\delta : v \rightarrow (-1)^{np+n+s} * d * v$ (Some authors define δ as the negative of this)

Properties: $\square \delta^2 = 0$

$\square$ In coordinates:

$$(\delta \omega)^{i_1 \dots i_{p-1}} = \frac{1}{\sqrt{|g|}} \left(\nabla_j \omega^{j i_1 \dots i_{p-1}} \right)_{,K}$$

$\square$ If M is contractible (and in every contractible part):

$$\delta v = 0 \Rightarrow \exists \omega : v = \delta \omega$$

Exercises: $\square$ Show the above.

$\square$ Determine whether or not δ is a derivation.

Use d and δ to obtain the Maxwell equations on M

$\square$ Define:

"Field strength": $F_{\mu\nu}(x) := \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$, $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$

↑ "Field strength" 2-form

↑ magn. field

electric field

"Current" 3-form $\rightarrow j(x) := \frac{1}{3!} \epsilon_{\mu\nu\lambda\sigma} j^\sigma dx^\mu \wedge dx^\nu \wedge dx^\lambda$

↑ "current 4-vector"

$\square$ Then: The Maxwell Eqs read:

"Homogeneous
Maxwell equations"
(indep. of metric)

$$dF = 0, \quad \delta F = *j$$

"Inhomogeneous
Maxwell equations"
(dependent on the metric)

Current 1-form, i.e., cotangent vector field

Remarks:

- F is assumed to be an exact 2-form, i.e.,:

$$F = dA$$

(the 1-form A is called the 4-potential)

- This already implies the homogeneous Maxwell equations:

$$dF = d^2A = 0 !$$

→ One calls them "structure equations".

- General relativity also possesses structure equations.

Remark:

The gauge principle of electrodynamics is the observation that, for any $w \in \Lambda_0$:

$$A \text{ and } \tilde{A} := A + dw$$

describe the same physics.

The Aharonov-Bohm effect and topological phases in general, can make A itself visible when Poincaré lemma doesn't apply.

They do because the (classically) observable fields are only the E and B fields in F and since $d^2 = 0$:

observable
 E and B fields

$$\rightarrow F = dA = d\tilde{A}$$

D The Laplacian/d'Alembertian, Δ , $\square$:

$\square$ Definition of the Laplacian:

$$\Delta := \delta d + d\delta$$

Some authors define Δ as the negative of this and let δ be the adjoint of d .

$\square$ Clear: $\Delta : \Lambda^p(M) \rightarrow \Lambda^p(M)$

$\square$ If signature $s=1$: Then also called d'Alembertian and denoted $\square := d\delta + \delta d$.

$\square$ Action on, e.g., $f \in \Lambda_0(M)$ in a chart: Exercise: verify

$$\square f = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{\mu\nu} f_{,\mu} \right)_{,\nu}$$

$$\left(= \left(-\frac{\partial^2}{\partial x^0{}^2} + \frac{\partial^2}{\partial x^1{}^2} + \frac{\partial^2}{\partial x^2{}^2} + \frac{\partial^2}{\partial x^3{}^2} \right) f \right) \text{ if } g = \eta$$

Properties of the d'Alembertian, $\square$ in the Hilbert space $\Lambda(M)$: , if Λ is a Hilbert space

* Defined: $\square : \Lambda_r(M) \rightarrow \Lambda_r(M)$

$$\square : \nu \rightarrow (\delta d + d\delta)\nu$$

* In the Hilbert space $\Lambda(M)$:

$$\square = \delta d + d\delta \text{ obeys } (d, \square \beta) = (\square d, \beta)$$

* $\square$ is self-adjoint, $\square = \square^*$, for suitable boundary conditions, or if $\partial M = \emptyset$ and assuming $(,)$ is positive definite.

* Exercises: $\square$ Verify $\square = \square^*$ formally, using only $\delta = -d^*$.

$\square$ Verify that $\square * = * \square$, $\square d = d \square$, $\square \delta = \delta \square$.

Consequences of the self-adjointness of Δ : - if M is a Hilbert space

- A) The operators Δ and $\square$ can be diagonalized, with real spectrum.
- B) For Riemannian manifolds, $\text{spec}(\Delta) \subset [0, \infty)$.
- C) For compact Riem. manifolds (of finite volume): $\text{spec}(\Delta)$ is discrete.
- D) Then, $\text{spec}(\Delta)$ is carrying a lot of information about (M, g) !
Still the finite volume Riemannian case.

Remark: There exists a related mathematical discipline, called "Spectral Geometry", combining differential geometry with functional analysis, i.e., the languages of general relativity and quantum theory!

of Riemannian manifolds →

Application: Klein-Gordon "action":

$$\begin{aligned}
 S[\phi] &::= \frac{1}{2} \int_M \overbrace{g^{\mu\nu}}^{\in T^2} \underbrace{\phi_{,\mu}}_{\in T_1} \underbrace{\phi_{,\nu}}_{\in T_1} \underbrace{\Omega}_{\in T_1} \\
 &\quad \uparrow \\
 &\quad \text{Klein Gordon field } \phi \in \mathcal{F}(M) \\
 &= \frac{1}{2} \int_M g^{\mu\nu}(x) \left(\frac{\partial}{\partial x^\mu} \phi \right) \left(\frac{\partial}{\partial x^\nu} \phi \right) \overbrace{\sqrt{|g(x)|} d^n x}^{\Omega} \\
 &\quad \uparrow \text{next: integrate by parts!} \\
 &= \frac{1}{2} \int_M -\phi \underbrace{\frac{\partial}{\partial x^\nu} \left(\sqrt{|g|} g^{\mu\nu} \frac{\partial}{\partial x^\mu} \phi \right)}_{=\square \phi} \underbrace{\frac{1}{\sqrt{|g|}} \sqrt{|g|} d^n x}_{=\Omega} \\
 &= -\frac{1}{2} \int \phi (\square \phi) \Omega
 \end{aligned}$$

(Recall special relativity: $S[\phi] = \int_{\mathbb{R}^4} \eta^{\mu\nu} \phi_{,\mu} \phi_{,\nu} d^4 x$)

Obtain the Klein Gordon wave equation:

□ Recall: Euler Lagrange equation $\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}$

□ Here: $\mathcal{L} = -\frac{1}{2} \phi \square \phi$ (the 0-form that we are integrating: $S' = \int_n \mathcal{L} \Omega$)

□ Obtain Klein Gordon equation:

$$\square \phi = 0$$

(with "mass": $\mathcal{L} = -\frac{1}{2} \phi (\square + m^2) \phi$
yielding $(\square + m^2) \phi = 0$)

Q: Which physical fields are described by K-G fields?

A: □ Meson fields

↙ there are many sorts of mesons. Most important mesons: "Pions". They transmit the nuclear force among protons & neutrons

□ Higgs field (Gives all particles their mass. Found at LHC. Nobel to Higgs, Englert (Brout) m2013)

□ Inflaton field (crucial ingredient in modern cosmology → see later)