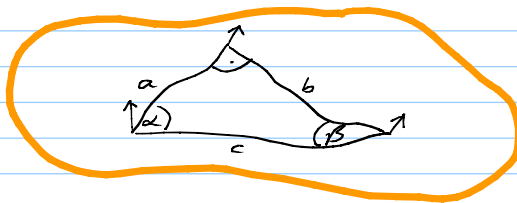


Recall: The nontrivial shape of a manifold reveals itself in several ways:



1. Violation of angle sum law, $\alpha + \beta + \gamma \neq 180^\circ$.

$\Rightarrow$ Can encode shape through deficit angles (used in some quantum gravity approaches)

2. Violation of Pythagoras' law, $a^2 + b^2 \neq c^2$.

$\Rightarrow$ Can encode shape through metric distances: (M, g)

3. Nontrivial parallel transport of vectors on loops.

$\Rightarrow$ Can encode shape through affine connection: (M, Γ)

This makes it hard to identify the true degrees of freedom, so that they can be quantized.

Observe: Such local descriptions carry redundant information!

Why? Two (pseudo-)Riemannian mflds $(M, g), (M, g')$ must be considered equivalent, i.e., they are describing the same space(-time), if there exists an isometric, i.e., metric-preserving, isomorphism:

$$\varphi: (M, g) \rightarrow (M, g')$$

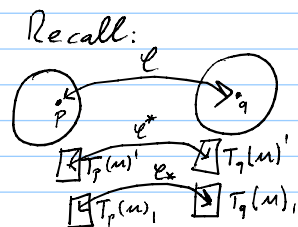
Here: φ is called metric-preserving if, under the pull-back map

$$T\varphi^*: T_p(M)_2 \rightarrow T_p(M)_2$$

the metric obeys:

$$T\varphi_*(g) = g'$$

$\Rightarrow \varphi$ can then be considered to be a mere change of chart.



Intuition: $(M, g), (M, g')$ that are related by an isometric diffeomorphism are mere cd changes of another, i.e., have the same "shape".

Definition: A (pseudo-) Riemannian structure, say Ξ , is an equivalence class of (pseudo-) Riemannian manifolds which can be mapped into each other via metric-preserving diffeomorphisms, i.e., via changes of coordinates.

$\Rightarrow$ Space(time) will need to be modelled as a (pseudo-) Riemannian structure, Ξ , i.e., as an equivalence class of pairs (M, g) .

Problem: These equiv. classes are hard to handle because absence or existence of $\mathcal{C}$ is hard to check!

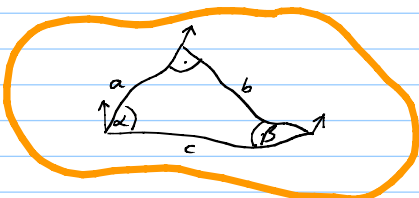
$\Rightarrow$ One would like to be able to reliably identify exactly one representative (M, g) per class Ξ .

□ This would be called a "fixing of gauge".

□ Why would this be useful?

A key example of when gauge fixing needed: Quantum gravity

We discussed detecting and describing shape through



- deficiency angles
- nontrivial metric distances (M, g)
- nontrivial parallel transport (M, Γ)

Recall: Quantum theory can be formulated in path integral form.

Applied to gravity:

Expect to have to handle path integrals of the type:

$$\int e^{iS(\Xi)} D\Xi$$

"all Riemannian structures Ξ "

But what we initially have is, roughly of the form:

$$\int e^{iS(g)} \delta(?) Dg \text{ or } \int e^{iS(\Gamma)} \delta(?) D\Gamma$$

"all g " "all Γ "

Here, $\delta(?)$ should be such that from each equivalence class of the g 's or the Γ 's only exactly one contributes to the path integral.

→ Much of Quantum Gravity research is concerned with working out suitable $\delta(?)$ for g 's or Γ 's or other variables formed from them, such as the frame fields (see "loop quantum gravity").

Q: Can one detect and describe a (pseudo-) Riemannian structure Ξ directly?

A: Possibly yes, using "Spectral Geometry":

Idea: A manifold's vibration spectrum $\{\lambda_n\}$ ^{Independent of coordinate systems!} depends only on Ξ !

Key question of the field of spectral geometry: (Weyl 1911)

Does the spectrum $\{\lambda_n\}$ encode all about the shape, i.e., Ξ ?

Remarks:

- It cannot, if $\mathcal{M}$ has infinite volume, because then the spectrum of Δ will become (almost) completely continuous.
- The spectral geometry of pseudo-Riemannian manifolds is still very little developed.

Theorem:

- Assume $(\mathcal{M}, g)$ is a compact Riemannian manifold without boundary, $\partial\mathcal{M} = \emptyset$.
↖ implies finite volume
- Then, each $\text{spec}(\Delta_p)$ is discrete, with finite degeneracies and without accumulation points.

In practice:

We can describe any arbitrarily large part of the universe by a compact Riemannian manifold, $(\mathcal{M}, g)$.

This allows us to describe, e.g., 3-dim. space at any fixed time (or also 4-dim. spacetime after so-called Wick rotation).

Types of waves (incl. sounds) on $\mathcal{M}$:

Consider p -form fields $w(x)$ on $\mathcal{M}$, with time evolution, e.g.,:
↖ assumed compact, no boundary

1. Schrödinger equation: $i\hbar \partial_t w(x, t) = -\frac{\hbar^2}{2m} \Delta_p w(x, t)$

2. Heat equation: $\partial_t w(x, t) = -\alpha \Delta_p w(x, t)$

3. Klein Gordon (and acoustic) eqn: $-\partial_t^2 w(x, t) = \beta \Delta_p w(x, t)$

□ Each of them can be solved via separation of variables:

□ Assume we find an eigenform $\tilde{\omega}(x)$ of Δ on $\mathcal{M}$:

$$\Delta_p \tilde{\omega}(x) = \lambda \tilde{\omega}(x)$$

□ They exist: Each Δ is self-adjoint, w.r.t. the inner product $(\omega, \nu) = \int_{\mathcal{M}} \omega \wedge \star \nu$.

Then: Schrödinger eqn solved by: $\omega(x, t) := e^{\frac{i\hbar}{2m}\lambda t} \tilde{\omega}(x)$

Heat eqn solved by: $\omega(x, t) := e^{-\frac{\lambda}{2m} t} \tilde{\omega}(x)$

Klein Gordon eqn solved by: $\omega(x, t) := e^{\pm i\sqrt{\lambda} t} \tilde{\omega}(x)$

⇒ The spectrum $\text{spec}(\Delta_p)$ is the overtone spectrum of p -form type waves on the manifold $\mathcal{M}$.

Properties of $\text{spec}(\Delta_p)$:

□ Expectations:

The spectra $\text{spec}(\Delta_p)$ for different p carry different information about $\mathcal{M}$:

E.g., scalar and vector seismic waves travel (and reflect) differently.

□ But recall also: a) $[\Delta, \star] = 0$

b) $[\Delta, d] = 0$

c) $[\Delta, \delta] = 0$

This will relate $\text{spec}(\Delta_p)$ to $\text{spec}(\Delta_{n-p})$, $\text{spec}(\Delta_{p+1})$ and $\text{spec}(\Delta_{p-1})$:

Use $[\Delta, *] = 0$:

Assume: $\omega \in \Lambda_p$ and $\Delta\omega = \lambda\omega$.

Define: $v := *\omega \in \Lambda_{n-p}$

Then:

$$\Delta v = \Delta*\omega = *\Delta\omega = *\lambda\omega = \lambda v$$

$$\Rightarrow \boxed{\text{spec}(\Delta_p) = \text{spec}(\Delta_{n-p})}$$

Next:

Careful utilization of $[\Delta, d] = 0$ and $[\Delta, \delta] = 0$ yields much more information about these spectra!

□ Notice that: Δ maps exact forms $\omega = dv$ into exact forms:

$$\Delta\omega = \Delta dv = \underbrace{d\Delta v}_{\text{an exact form}}$$

i.e.:

$$\boxed{\Delta: d\Lambda_r \rightarrow d\Lambda_r}$$

$d\Lambda_r = \text{image of } \Lambda_r \text{ under } d.$

□ Analogously: Δ maps co-exact forms $\omega = \delta\beta$ into co-exact forms:

$$\Delta\omega = \Delta\delta\beta = \underbrace{\delta\Delta\beta}_{\text{a co-exact form}}$$

i.e.:

$$\boxed{\Delta: \delta\Lambda_r \rightarrow \delta\Lambda_r}$$

□ Also: Δ can map forms into 0, namely its eigenspace with eigenvalue 0, denoted Λ_r^0 . Λ_r^0 is called the space of "harmonic" p -forms.

$$\boxed{\Delta: \Lambda_r^0 \rightarrow 0}$$

Thus: Δ maps $d\Lambda_r$ and $\delta\Lambda_r$ and Λ_r^0 into themselves.

Are there any other forms that Δ could act on? No!

Proposition ("Hodge decomposition"):

$$\Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^0$$

(Recall that $\oplus$ implies that the three spaces are orthogonal!)

Q: Why useful?

A: It means that every eigenvector of Δ_p is either in $d\Lambda_{p-1}$, or in $\delta\Lambda_{p+1}$, or in Λ_p^0 but is never a linear combination of vectors in these spaces.

Proof: It is clear that $d\Lambda_{p-1} \subset \Lambda_p$ and $\delta\Lambda_{p+1} \subset \Lambda_p$.

We need to show the orthogonalities and completeness:

□ Show that $d\Lambda_{p-1} \perp \delta\Lambda_{p+1}$:

Indeed, assume $\omega = dv \in \Lambda_p$ and $\alpha = \delta\beta \in \Lambda_p$.

$$\text{Then: } (\omega, \alpha) = (dv, \delta\beta) \stackrel{\substack{\text{use} \\ -d^*\delta}}{=} (d\overset{0}{d}v, \beta) = 0 \quad \checkmark$$

Exercise:
study the
remainder
of the proof.

□ Show that if $\omega \in \Lambda_p$ and $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$ then: $\omega \in \Lambda_p^0$.

Indeed, assume $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$. Then:

$$\forall \alpha: (d\alpha, \omega) = 0 \quad \text{i.e.} \quad -(\alpha, \delta\omega) = 0 \Rightarrow \delta\omega = 0$$

$$\forall \beta: (\delta\beta, \omega) = 0 \quad \text{i.e.} \quad -(\beta, d\omega) = 0 \Rightarrow d\omega = 0$$

$$\Rightarrow \Delta\omega = (d\delta + \delta d)\omega = 0 \Rightarrow \omega \in \Lambda_p^0 \quad \checkmark$$

□ Show that if $\omega \in \Lambda_p^\circ$ then $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$.

Assume $\omega \in \Lambda_p^\circ$, i.e., $\Delta\omega = 0$, i.e., $(\delta d + d\delta)\omega = 0$.

$$\Rightarrow (\omega, (d\delta + \delta d)\omega) = 0$$

$$\Rightarrow \overbrace{(\delta\omega, \delta\omega)}^{\geq 0} + \overbrace{(d\omega, d\omega)}^{\geq 0} = 0 \Rightarrow \delta\omega = 0 \text{ and } d\omega = 0.$$

(I.e., harmonic forms are closed and co-closed but not exact or co-exact.
Thus, $B_p := \dim(\Lambda_p^\circ)$ measures topological nontriviality.
The B_p are called the "Betti numbers".)

$$\Rightarrow \forall \alpha \in \Lambda_{p-1}: (\alpha, \delta\omega) = 0, \text{ i.e., } (d\alpha, \omega) = 0.$$

$$\Rightarrow \omega \perp d\Lambda_{p-1} \quad \checkmark$$

$$\text{Also: } \forall \beta \in \Lambda_{p+1}: (\beta, d\omega) = 0, \text{ i.e., } (\delta\beta, \omega) = 0.$$

$$\Rightarrow \omega \perp \delta\Lambda_{p+1} \quad \checkmark$$

Conclusion so far:

In the Hodge decomposition,
 Δ maps every term into
itself, i.e., Δ can be diagonalized
in each $d\Lambda_r$, $\delta\Lambda_r$, Λ_r° separately.

$$\begin{cases} \vdots \\ \Lambda_{p-1} = d\Lambda_{p-2} \oplus \delta\Lambda_p \oplus \Lambda_{p-1}^\circ \\ \Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^\circ \\ \Lambda_{p+1} = d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \Lambda_{p+1}^\circ \\ \vdots \end{cases}$$

$\Rightarrow \Delta$ has eigenvectors and -values on each of these subspaces, for all r :

$$\text{spec}(\Delta|_{d\Lambda_r}), \text{spec}(\Delta|_{\delta\Lambda_r}), \text{spec}(\Delta|_{\Lambda_r^\circ}) = \{0\} \dots$$

These spectra are related!

Proposition:

$$\text{spec}(\Delta|_{d\Lambda_r}) = \text{spec}(\Delta|_{\delta\Lambda_{r+1}})$$

and for each eigenvector in one there is one in the other.

This means:

$$\begin{aligned} & \vdots \\ \Lambda_{p-1} &= d\Lambda_{p-2} \oplus \underbrace{\delta\Lambda_p}_{\text{same spectrum}} \oplus \Lambda_{p-1}^\circ \\ \Lambda_p &= d\Lambda_{p-1} \oplus \underbrace{\delta\Lambda_{p+1}}_{\text{same spectrum}} \oplus \Lambda_p^\circ \\ \Lambda_{p+1} &= d\Lambda_p \oplus \underbrace{\delta\Lambda_{p+2}}_{\text{same spectrum}} \oplus \Lambda_{p+1}^\circ \\ & \vdots \end{aligned}$$

Proof:

Assume: $\lambda \in \text{spec}(\Delta|_{d\Lambda_r})$ with eigenvector $w \in d\Lambda_r$.

Define: $v := \delta w \in \delta\Lambda_{r+1}$

Then: $\Delta v = \Delta \delta w = \delta \Delta w = \lambda \delta w = \lambda v$

$\Rightarrow \lambda \in \text{spec}(\Delta|_{\delta\Lambda_{r+1}})$ and v is the eigenvector.

Conversely:

Assume: $\lambda \in \text{spec}(\Delta|_{\delta\Lambda_{r+1}})$ with eigenvector $w \in \delta\Lambda_{r+1}$.

Define: $v := dw \in d\Lambda_r$

Then: $\Delta v = \Delta dw = d\Delta w = \lambda dw = \lambda v$

$\Rightarrow \lambda \in \text{spec}(\Delta|_{d\Lambda_r})$ and v is the eigenvector.



Re-use $[\Delta, *] = 0$:

Proposition: $*$: $d\Lambda_r \rightarrow \delta\Lambda_{n-r}$

i.e.: $*$: exact $r+1$ forms $\rightarrow$ co-exact $n-r-1$ forms

Proof: Assume $\omega = d\zeta \in d\Lambda_r$

Define $v := *\omega$

$$\begin{aligned} \Rightarrow v &= *d\zeta = (-1)^{r(n-r)} \underbrace{*\delta}_{\parallel} d*\zeta \\ &= \delta\alpha \in \delta\Lambda_{n-r} \text{ for } \alpha = (-1)^{r(n-r)} *\zeta \end{aligned}$$

Proposition: $*$: $\delta\Lambda_r \rightarrow d\Lambda_{n-r}$

Proof: Exercise.

Recall: $*$ preserves the spectrum of Δ as we showed already.

$\Rightarrow$

Summary:

$$\begin{aligned} \Lambda_{p-1} &= d\Lambda_{p-2} \oplus \delta\Lambda_p \oplus \Lambda_{p-1}^0 \\ \Lambda_p &= d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^0 \\ \Lambda_{p+1} &= d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \Lambda_{p+1}^0 \\ &\vdots \end{aligned}$$

(brackets under $d\Lambda_{p-2}$ and $\delta\Lambda_p$ in the first row, and $d\Lambda_{p-1}$ and $\delta\Lambda_{p+1}$ in the second row, are labeled "same spectrum")

Now we also found:

$$\begin{aligned} \Lambda_p &= d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^0 \\ &\vdots \\ \Lambda_{n-p} &= d\Lambda_{n-p-1} \oplus \delta\Lambda_{n-p+1} \oplus \Lambda_{n-p}^0 \end{aligned}$$

(brackets under $d\Lambda_{p-1}$ and $\delta\Lambda_{p+1}$ in the first row, and $d\Lambda_{n-p-1}$ and $\delta\Lambda_{n-p+1}$ in the last row, are labeled "same spectrum")

Example: $\dim(\mathcal{M})=3$

Exercise: do same for $\dim(\mathcal{M})=4$

$$\Lambda_0 = \delta \Lambda_1 \oplus \Lambda_0^\circ$$

$$\Lambda_1 = d \Lambda_0 \oplus \delta \Lambda_2 \oplus \Lambda_1^\circ$$

$$\Lambda_2 = d \Lambda_1 \oplus \delta \Lambda_3 \oplus \Lambda_2^\circ$$

$$\Lambda_3 = d \Lambda_2 \oplus \Lambda_3^\circ$$

Same color means same spectrum of Δ .

Conclusion: There is relatively little independent information in the spectra of p -form waves on $\mathcal{M}$!
E.g., when $\dim(\mathcal{M})=3$, then the spectrum of co-vector waves $\text{spec}(\Delta|_{\Lambda_1})$ has already all information of all these spectra.

Literature: (neglecting literature on detecting boundary shapes from spectra)

Indeed: The spectra of Δ do not contain sufficient information in general to uniquely identify the Riemannian structure from the spectra alone:

Examples: Cases have been found of pairs $(\mathcal{M}, g), (\tilde{\mathcal{M}}, \tilde{g})$ that are isospectral for Δ on all Λ_p but that are not diffeomorphically isometric!

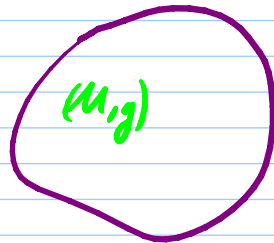
Nevertheless: All examples are of limited significance:

- manifolds that are locally, if not globally isometric, or
- manifolds that are isospectral only w. respect to some Δ or
- manifolds that are discrete pairs (e.g. mirror images).

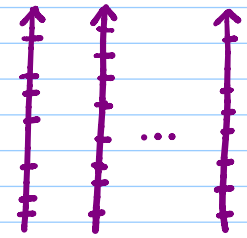
Fresh approach to spectral geometry (AK)

Strategy: Iterate infinitesimal inverse spectral geometry

Assume both, the mfd and its spectra are given:



A compact Riemannian manifold (M, g) without boundary



The spectra $\{\lambda_n^{(i)}\}$ of Laplacians $\Delta^{(i)}$ on the manifold.

↑
Could be Laplacians not only on forms but also on general tensors.

Perturbation:

Now change the shape of (M, g) slightly, through:

$$g \rightarrow g + h$$

This will slightly change the spectra to

$$\{\lambda_n^{(i)}\} \rightarrow \{\lambda_n^{(i)} + \mu_n^{(i)}\}$$

Why is this linearization useful?

□ One can define a self-adjoint Laplacian $\Delta^{(m)}$ on $T_2(M)$, with Hilbert basis $\{b_m(x)\}$ and eigenvalues $\{\lambda_n^{(m)}\}$:

$$\Delta^{(m)} b_m(x) = \lambda_n b_m(x)$$

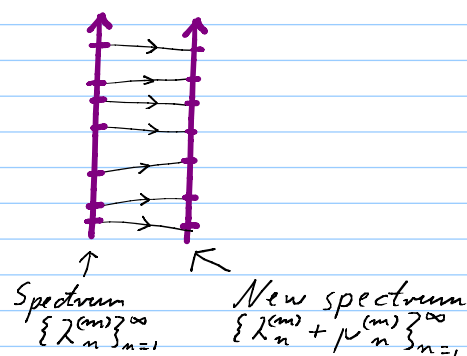
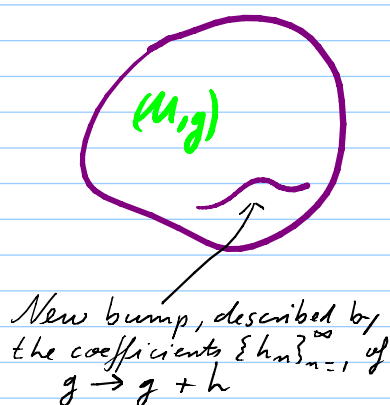
⇒ The metric's perturbation $h \in T_2(\mathcal{M})$ can be expanded:

$$h = \sum_{n=1}^{\infty} h_n b_n(x)$$

The perturbation of $\text{spec}(\Delta^{(m)})$ is:

$$\{\lambda_n^{(m)}\} \rightarrow \{\lambda_n^{(m)} + \mu_n^{(m)}\}$$

⇒



⇒ We obtain a linear map S :

$$S: \{h_n\} \rightarrow \{\mu_n\}$$

$$S: h_n \rightarrow \mu_n = S_{nm} h_m$$

Notice:

Consider only eigenvectors and eigenvalues up to a cutoff scale. Then, there are as many parameters $\{h_n\}_{n=1}^N$ as $\{\mu_n\}_{n=1}^N$.

⇒ S is a square matrix.

∩ $\det(S) \neq 0$, then S^{-1} exists.

⇒ should be able to iterate the perturbations?

This is ongoing research.

Remarks: $\square$ Not all h actually change the shape:

Iff $h = L_{\xi}g$ for some vector field ξ , then $g \rightarrow g + h$ is merely the infinitesimal change of chart belonging to the flow induced by ξ .

$\square$ Symmetric covariant 2-tensors such as h have a canonical decomposition similar to the Hodge decomposition. Thus, Δ has three spectra on $T_2(M)$.

Reference: See also e.g. the video of my talk at PI: <http://pirsa.org/15090062>

Infinitesimal spectral geometry arose from my paper on how Spacetime could be simultaneously continuous and discrete, in the same way that information can.