

Plan: **I** The dynamics of matter & radiation in curved spacetime

**II** Energy - momentum tensor

**III** The dynamics of spacetime itself.

1. Recall: On a (pseudo)-Riemannian mfd, equations are well-defined only if defined independently of any chart.

⇒ Any eqn, including the eqns of motions for matter fields must be eqns among tensors and their covariant derivatives.

⇒ Need a tensor field,  $\psi$ , for each species of particle:

$e^-$ ,  $q$ , gluon,  $\pi^\pm$ , photon,  $W^\pm$ , etc...

Notation:

$\psi_{(i)}^{a\dots b}$  contravariant  
 $c\dots d$  covariant  
↑ species label

Note: any spinor equation can also

be expressed as a (complicated) tensor equation

(see e.g. Hawking & Ellis, p 59)

Question:

Could we have also an additional connection field  $\tilde{\Gamma}_{ij}^k$ ?

Yes, we could: But, the difference field  $Q^k_{ij} := \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}$  is actually a tensor field!

$$\Gamma^r_{ab} \rightarrow \frac{\partial \bar{x}^r}{\partial x^a} \frac{\partial^2 x^i}{\partial \bar{x}^a \partial x^b} + \frac{\partial \bar{x}^r}{\partial x^a} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij}$$

$$\tilde{\Gamma}^r_{ab} \rightarrow \frac{\partial \bar{x}^r}{\partial x^a} \frac{\partial^2 x^i}{\partial \bar{x}^a \partial x^b} + \frac{\partial \bar{x}^r}{\partial x^a} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \tilde{\Gamma}^k_{ij}$$

$$\Rightarrow (\Gamma^r_{ab} - \tilde{\Gamma}^r_{ab}) \rightarrow \frac{\partial \bar{x}^r}{\partial x^a} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} (\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij})$$

$$\Rightarrow \boxed{Q^r_{ab} \rightarrow \frac{\partial \bar{x}^r}{\partial x^a} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} Q^r_{ab}}$$

i.e.  $Q^r_{ab}$  is a tensor due to having the correct transformation property according to the physicist's definition of a tensor.

$\Rightarrow$  Introducing an additional connection  $\tilde{\Gamma}$  is same as introducing simply a new tensor field  $Q$ .

Remark:  $\Rightarrow$  "variations"  $\delta \Gamma^r_{ab}$  will behave tensorially!

Eqs of motion of matter fields?

Action principle: (As in special relativity)

Any theory of matter fields can be defined by specifying the so-called Lagrangian function,  $L$ , namely a scalar function of the matter fields  $\Psi_{(i)}^{a \dots b} \dots c \dots d$  and their first covariant derivatives, and now also of the metric  $g$ :

[we'll sometimes omit the indices]

$$\boxed{L(\Psi) = L^{(matter)}(\{\Psi_{(i)}^{a \dots b} \dots c \dots d\}, \{\Psi_{(i)}^{a \dots b} \dots c \dots d; e\}, g)}$$

□ Define the action functional:

$$S[\psi] := \int_B \underbrace{L(\psi)}_{\text{scalar}} \underbrace{\sqrt{g}}_{\text{n-form}} d^4x \quad \in \mathbb{R}$$

$\Omega = \text{volume form}$   
 some bounded and closed 4-dim region in  $M$ .

Thus, each physical field  $\psi(x,t)$  (as a function of both space and time) is mapped into a number  $S[\psi]$ .

□ Action principle (or postulate) of classical physics:

In nature, physical fields  $\psi$  are such that  $S[\psi]$  is extremal in the space of all fields  $\psi$ .

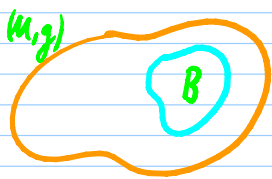
□ Thus: The matter fields  $\psi$  obey:

$$\boxed{\frac{\delta S[\psi]}{\delta \psi} = 0} \quad (*)$$

These will be the eqns of motion for the fields  $\psi$ .

□ Definition of (\*)?

Def: A "variation  $\delta \psi$ " of the fields  $\psi_i(p)$  in a region  $B$  is a one-parameter deformation,  $\psi_i(\lambda, p)$ , with  $\lambda \in (-\epsilon, \epsilon)$ , some finite interval  
 $\downarrow$  deformation parameter  
 $p \in B \subset M$



so that

- 1.)  $\Psi_{(i)}(0, p) = \Psi_{(i)}(p) \quad \forall p \in M$   
*i.e.  $\lambda=0$  is non-deformation*
- 2.)  $\Psi_{(i)}(\lambda, p) = \Psi_{(i)}(p) \quad \forall \lambda, \text{ if } p \in M - B$   
*i.e. no deformation at all outside region B.*

Def: Then, we define:

$$\delta \Psi_{(i)}(p) := \left. \frac{\partial \Psi_{(i)}(\lambda, p)}{\partial \lambda} \right|_{\lambda=0}$$

Def: The action principle now reads:

$$0 = \left. \frac{\partial S[\Psi]}{\partial \lambda} \right|_{\lambda=0} \quad \text{for all variations } \delta \Psi_{(i)}.$$

Evaluate:

$$0 = \left. \frac{\partial S}{\partial \lambda} \right|_{\lambda=0} = \sum_i \int_B \left[ \overbrace{\frac{\partial \mathcal{L}}{\partial \Psi_{(i)}^{a \dots b} c \dots d} \delta \Psi_{(i)}^{a \dots b} c \dots d}^{\text{Term I}} + \underbrace{\frac{\partial \mathcal{L}}{\partial \Psi_{(i)}^{a \dots b} c \dots d j e}}_{\substack{\text{by assumption,} \\ \mathcal{L} \text{ depends also on} \\ \text{the 1st cov. derivatives}}} \delta(\Psi_{(i)}^{a \dots b} c \dots d j e) \right] \sqrt{g} d^4 x$$

Term II

Evaluate terms I, II separately:

## Term II:

□ We notice:

Recall: At origin of geodesic coordinate system,  $\Gamma^k_{ij} = 0$ , i.e.  $\psi_{,e} = \psi_{,e}$ . But then  $\frac{\partial}{\partial x^i}$  and  $\frac{\partial}{\partial \lambda}$  commute. True in any coordinate system.

$$\delta(\psi_{(i)}^{a \dots b} \dots_{,j,e}) = (\delta \psi_{(i)}^{a \dots b} \dots_{,d})_{,j,e}$$

$$\Rightarrow \text{Term II} = \sum_i \int_B \frac{\partial \mathcal{L}}{\partial \psi_{(i)}^{a \dots b} \dots_{,j,e}} (\delta \psi_{(i)}^{a \dots b} \dots_{,d})_{,j,e} \sqrt{g} d^4 x$$

$$= \sum_i \int_B \left[ \left( \frac{\partial \mathcal{L}}{\partial \psi_{(i)}^{a \dots b} \dots_{,j,e}} \delta \psi_{(i)}^{a \dots b} \dots_{,d} \right)_{,j,e} - \left( \frac{\partial \mathcal{L}}{\partial \psi_{(i)}^{a \dots b} \dots_{,j,e}} \right)_{,j,e} \delta \psi_{(i)}^{a \dots b} \dots_{,d} \right] \sqrt{g} d^4 x$$

(use Leibniz rule to verify)

One term is a "boundary term":

$$\begin{aligned} & \sum_i \int_B K^e_{,j,e} \sqrt{g} d^4 x \\ &= \sum_i \int_B \text{div}_\Omega K \end{aligned}$$

Exercise:

show that for all  $\xi^\mu$ :

$$\xi^\mu_{; \mu} \Omega = \text{div}_\Omega \xi$$

$$\text{if } \Omega = \sqrt{g} dx^1 \wedge \dots \wedge dx^m$$

Gauß' theorem  $\Rightarrow$

$$= \sum_i \int_{\partial B} i_K \Omega$$

← inner derivation

$$\left( \begin{aligned} \text{Recall: } \text{div}_\Omega K &= L_K \Omega \\ &= (i_K d + d i_K) \Omega \\ &= d i_K \Omega \end{aligned} \right)$$

but:  $K \propto \delta \psi$  and  $\delta \psi(p) = 0$  if  $p \in \partial B$

by property 2) of variations.

$$\Rightarrow = 0 !$$

Thus, term II simplifies and we obtain:

$$0 = \frac{\partial S}{\partial \lambda} \Big|_{\lambda=0} = \sum_i \int_B \left[ \overbrace{\frac{\partial \mathcal{L}}{\partial \Psi_{(i)}^{a \dots b \dots d}} \delta \Psi_{(i)}^{a \dots b \dots d}}^{\text{Term I}} - \overbrace{\left( \frac{\partial \mathcal{L}}{\partial \Psi_{(i)}^{a \dots b \dots d; e}} \right)_{; e} \delta \Psi_{(i)}^{a \dots b \dots d}}^{\text{Term II}} \right] \sqrt{g} d^4 x$$

Since must hold for all variations  $\delta \Psi$

$\Rightarrow$

$$\frac{\partial \mathcal{L}}{\partial \Psi_{(i)}^{a \dots b \dots d}} - \left( \frac{\partial \mathcal{L}}{\partial \Psi_{(i)}^{a \dots b \dots d; e}} \right)_{; e} = 0$$

"Euler-Lagrange equations"

Given  $\mathcal{L}(\Psi)$ , these eqns yield the eqns. of motion for  $\Psi$ .

Example: Areal-valued scalar field  $\Psi$   $\swarrow$  real-valued

▢ Such  $\Psi$  describe e.g.:

- $\pi^0$  meson (quark + anti-quark)
- inflaton

▢ Lagrangian?

- Choose geodesic cds at orb. point and appeal to equiv. principle.
- Obtain from spec. relat. Lagrangian:

$$\mathcal{L} = -\frac{1}{2} \left( \Psi_{;a} \Psi_{;b} g^{ab} + \frac{m^2}{\hbar^2} \Psi^2 \right)$$

▢ Euler-Lagrange equation: Klein-Gordon equation

(Exercise: verify)

$$\Psi_{;ab} g^{ab} - \frac{m^2}{\hbar^2} \Psi = 0$$

### Example: The electromagnetic fields

- Assume there are no charges (i.e. there are only EM waves)
- Define the "EM 4-potential" as a real-number-valued one-form  $A$ .
- Consider the field strength tensor  $F$ :

$$F := dA$$

- Recall that the  $E$  and  $B$  fields are components of the 2-form  $F$ . (up to a factor of 2)

- The Lagrangian (from equiv. principle):

$$L = -\frac{1}{16\pi} F_{ab} F_{cd} g^{ac} g^{bd} \quad \left( \text{Exercise: write in terms of forms} \right)$$

- Varying w. resp. to  $A$ , the E.L. equations read:

$$F_{ab;c} g^{bc} = 0$$

recall: this is  $\delta F = 0$

- It is also true that

$$F_{ab;c} + F_{ca;b} + F_{bc;a} = 0$$

"Maxwell eqns".

but this is not an Euler Lagrange eqn. It

is simply:

$$dF = 0$$

(which holds because  $F = dA$  and  $d^2 = 0$ )

Example: A charged scalar field  $\psi$ , <sup>← complex-valued</sup>  
 (such  $\psi$  describe, e.g.,  $\pi^\pm$  mesons)  
 together with electromagnetism.

□ Equiv. principle yields from spec. relativity:

Why  $\psi$  complex?  
 Mixed term is Lorentz force  
 If  $\psi$  was real, it would be  
 absent:  
 $-ie A_n \psi^* \psi_{,n} g^{ab}$   
 $+ ie A_b \psi_{,a} \psi g^{ab}$   
 $= ie A_n g^{ab} (\psi^*_{,a} \psi_{,b} - \psi_{,a} \psi^*_{,b})$   
 $= 0$  if  $\psi^* = \psi$

$$L = -\frac{1}{2} (\psi_{,a}^* - ie A_a \psi^*) (\psi_{,b} + ie A_b \psi) g^{ab}$$

electric charge constant

$$= -\frac{1}{2} \frac{m^2}{\hbar^2} \psi^* \psi - \frac{1}{16\pi} F_{ab} F_{cd} g^{ac} g^{bd}$$

□ Vary w. resp. to  $\psi^* \Rightarrow$  E.L. eqn:

$$\underbrace{\psi_{,a;b} g^{ab} - \frac{m^2}{\hbar^2} \psi}_{\text{Klein Gordon part}} + \underbrace{ie A_a g^{ab} (\psi_{,b} + ie A_b \psi) + ie A_{a;b} g^{ab} \psi}_{\psi \text{ is affected by } A} = 0$$

and varying w. resp. to  $\psi$  yields the compl. conj. equation.

□ Vary w. resp. to  $A_a \Rightarrow$  E.L. eqn:

$$\underbrace{\frac{1}{4\pi} F_{ab;c} g^{bc}}_{\text{plain Maxwell part}} - \underbrace{ie \psi (\psi_{,a}^* - ie A_a \psi^*) + ie \psi^* (\psi_{,a} + ie A_a \psi)}_{A \text{ is affected by } \psi, \psi^*} = 0$$

Dirac equation: (Brief treatment of basics only of Dirac spinors)

In special relativity: (with units such that  $\hbar=1$ )

$$\boxed{\left(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m\right) \Psi(x) = 0} \quad \text{"Dirac equation"} \quad (D)$$

where  $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$  is a "Spinor"  
↑ describes spin  $1/2$  particles such as electrons and quarks

and the four  $4 \times 4$  matrices  $\gamma^\mu$  obey:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu} \quad (*)$$

↖  $\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{\mu\nu}$

□ Why (\*)? Equation (\*) is specifically chosen so that each component of  $\Psi$  obeys the Klein Gordon equation. Indeed:

$$(D) \Rightarrow (-i \gamma^\mu \partial_\mu - m)(i \gamma^\nu \partial_\nu - m) \Psi = 0$$

$$\Rightarrow (+\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + i \gamma^\mu \partial_\mu m - i m \gamma^\nu \partial_\nu + m^2) \Psi = 0$$

$$\Rightarrow \underbrace{(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)}_{\text{symmetric under } \mu \leftrightarrow \nu} \Psi = 0$$

antisymmetric part not needed, it would drop out.

$$\Rightarrow \left(\frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu + m^2\right) \Psi = 0$$

$$\stackrel{(*)}{\Rightarrow} \quad \underline{1} \quad (\eta^{\mu\nu} \partial_\mu \partial_\nu + m^2) \Psi = 0$$

which is the Klein Gordon equation in flat space.

In general relativity:

□ By choosing an orthonormal tetrad,  $\{e^i\}$ , we achieve

$$g^{\mu\nu} = \eta^{\mu\nu} \quad \forall p \in M$$

i.e. one set of matrices  $\gamma^\mu$  obeying  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$  suffices.

□ This motivates:

$$(i \gamma^\mu \nabla_\mu - m) \Psi = 0$$

□ But what is the covariant derivative of a spinor?

$$\nabla_{e_\mu} \Psi = ?$$

Recall: The covariant derivative of a vector yields the infinitesimal Lorentz transformation by which the vector rotates under infinitesimal parallel transport.

Idea: The covariant derivative of a spinor should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

Recall: Infinitesimal parallel transport of a vector  $e_\beta$  in direction  $e_\mu$ :

$$e_\beta \rightarrow e_\beta + \nabla_{e_\mu} e_\beta = e_\beta + \omega_\beta^\alpha(e_\mu) e_\alpha$$

Recall: the curvature 2-form takes values that are infinitesimal Lorentz transformations.

(Recall intuition why parallel transport yields Lorentz transformation: Parallel transport preserves the lengths of vectors, i.e. they can at most "rotate" and in 3+1 dim. this is Lorentz transformations.)

This is an infinitesimal Lorentz transformation  $\Lambda_\beta^\alpha$ :

$$e_\beta \rightarrow \Lambda_\beta^\alpha e_\alpha \quad \text{with} \quad \Lambda_\beta^\alpha = \delta_\beta^\alpha + \omega_\beta^\alpha(e_\mu)$$

because  $\omega_\beta^\alpha$  obeys:  $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ . (Which is the defining equation for infinitesimal Lorentz transformations)

Now that we know the inf. Lorentz transf. for any inf. parallel transport:

→ Strategy: Apply the same inf. Lorentz transformation on spinors for their parallel transport.

To this end: Recall from Special Relativity how an infinitesimal Lorentz transformation acts on a spinor:

□ Assume  $\{s_i\}_{i=1}^4$  are ON basis in Spinor space, i.e.

$$\psi = \psi^i(x) s_i$$

these are Spinor indices:  $i = 1, 2, 3, 4$

□ How do the  $s_i$  transform under Lorentz transformations? I.e., what is  $\nabla_{e_i} s_j = ?$  (In analogy to  $\nabla_{e_a} e_\mu = \omega^\nu_{\mu a} e_\nu$ )

□ From special relativity it is known that under infinitesimal Lorentz transformations,

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$$

vectors transform as

$$e_\mu \rightarrow e_\mu + \omega^\nu_\mu e_\nu$$

and the Dirac spinors transform as:

$$s_i \rightarrow s_i - \frac{1}{4} \omega^\mu_\nu [x^\mu, \gamma^\nu] s_i$$

⇒ under infinitesimal Lorentz transf. the spinor "rotates" by this amount.

Where does  $[x^\mu, \gamma^\nu]$  come from?

Recall that e.g. translations in space are generated by momentum operators,  $e^{i\vec{p}\cdot\vec{x}} f(x) e^{-i\vec{p}\cdot\vec{x}} = f(x+\vec{x})$ , if they obey the commutation relations  $[x_i, p_j] = i\delta_{ij}$ .

Similarly, Lorentz transformations are generated by operators  $M^{\mu\nu}$ :  $e^{i\omega_{\mu\nu} M^{\mu\nu}} f e^{-i\omega_{\mu\nu} M^{\mu\nu}} = \Lambda(f)$  if these  $M^{\mu\nu}$  obey certain commutation relations. In spinor space, the unique objects that obey these commutation relations are the  $M^{\mu\nu} = [x^\mu, \gamma^\nu]$ .

### Apply to GR:

If a vector  $e_\mu$  is infinitesimally parallel transported in the direction of  $e_a$  then it obtains an infinitesimal "rotation", namely the infinitesimal Lorentz transformation

$$\omega^\nu_\mu(e_a)$$

which is the value of the connection 1-form, i.e.:

local value of the connection form

$$e_\mu \rightarrow e_\mu + \overbrace{\omega^\nu_\mu(e_a)} e_\nu$$

→ From this one can immediately read off again the covariant derivative for vectors:

$$\nabla_a e_\mu = \omega^\nu_\mu(e_a) e_\nu$$

□ Now, when a spinor  $s_i$  is infinitesimally parallel transported in the direction of  $e_a$  then it too experiences the infinitesimal rotation, i.e., the infinitesimal Lorentz transformation

$$\omega^\nu_\mu(e_a)$$

which is the value of the connection 1-form. Thus:

local infinitesimal Lorentz transformation,  
i.e., local value of the connection 1-form.

$$s_i \rightarrow s_i - \frac{1}{4} \overbrace{\omega(e_a)_\mu}^{\omega^\nu_\mu(e_a)} [\gamma^\mu, \gamma^\nu] s_i$$

□ Since, under infinitesimal parallel transport:

$$s_i \rightarrow s_i + \nabla_a s_i$$

to be determined

⇒ The covariant derivative of the basis vectors  $\{s_i\}$  of Dirac spinors is:

$$\nabla_{e_a} s_i = -\frac{1}{4} \omega_{\mu}^{\nu}(e_a) [\gamma^{\mu}, \gamma^{\nu}] s_i$$

⇒ For general Dirac spinors  $\Psi(x) = \Psi^i(x) s_i$  the Leibniz rule for  $\nabla$  yields:

$$\nabla_{e_a} \Psi = \nabla_{e_a} (\overset{\text{scalar coefficient functions}}{\Psi^i(x)} s_i) = (\nabla_{e_a} \Psi^i(x)) s_i + \Psi^i(x) \nabla_{e_a} s_i$$

i.e.:

$$\nabla_{e_a} \Psi = e_a(\Psi) - \frac{1}{4} \omega(e_a)_{\mu}^{\nu} [\gamma^{\mu}, \gamma^{\nu}] \Psi$$

$$\uparrow \quad \quad \quad \uparrow \text{function} \\ e_a(\Psi) = s_i \underbrace{e_a(\Psi^i)}_{\text{vector field}}$$

Dirac equation:

The general relativistic Dirac equation

$$(i \gamma^{\mu} \nabla_{e_{\mu}} - m) \Psi = 0$$

now takes this explicit form:

$$i \gamma^{\mu} e_{\mu}(\Psi) - i \frac{1}{4} \omega(e_{\mu})_{\nu}^{\rho} \gamma^{\mu} [\gamma^{\nu}, \gamma^{\rho}] \Psi - m \Psi = 0$$

in a chart, this becomes a directional derivative of  $\Psi$ .

**Remark:** The relationship between the Dirac operator  $D = i \gamma^{\mu} \nabla_{e_{\mu}}$  and the Laplace or d'Alembert operator  $\square$  also becomes:

$$D = d + \delta.$$

To this end, one re-interprets the Grassmann algebra of differential forms as a so-called **Clifford algebra**.