

Friedmann-Lemaître cosmological solutions

Experimental evidence:

Hubble, Humason 1929

- The universe is and has been expanding.
- It appears to be spatially essentially isotropic and homogeneous on scales larger than a few (3-4) billion light years.

↑
(see e.g. Sloan Digital Sky Survey (SDSS)
at www.sdss.org)

Idealizing models:

- Assume perfect spatial isotropy and homogeneity:
- $\Rightarrow$ "Friedmann & Lemaître" (later Robertson & Walker) spacetimes

Concretely:

We assume we can model spacetime as a manifold (M, g) with:

$$M = I \times \Sigma$$

$$g = -dt^2 + a^2(t) \bar{g}$$

↑ In the basis $\{dx^\mu\}$ which comes with the coordinate system.

(we will later use an ON frame so that $g_{\mu\nu} = \eta_{\mu\nu}$)

Here:

- I is an interval, $I \subset \mathbb{R}$, and $t \in I$ is called "cosmic time". $a(t)$ is called the "scale factor".
- $(\Sigma, \bar{g})$ is a fully isotropic & homogeneous Riemannian manifold, i.e., a manifold of constant curvature, providing a 3-dim. surface of homogeneity at each point in cosmic time.

What are the possible Riemannian manifolds of constant curvature?

□ The Riemann tensor $\bar{R}_{ij,kl}$ must be expressible in terms of a constant, say K , which fixes the curvature's strength, and the tensorial part can only depend on the metric $\bar{g}$.

⇒ Given the index symmetries of $\bar{R}_{ij,kl}$ it should (and does) take the form:

$$\bar{R}_{ij,kl} = K (\bar{g}_{ik} \bar{g}_{jl} - \bar{g}_{il} \bar{g}_{jk}) \quad (*)$$

$$\Rightarrow \bar{R}_{je} = 2K \bar{g}_{je}, \bar{R} = 6K$$

⇒ Using a "Triad" $\{\bar{\theta}^i\}$:

(ON bases of $T_p(\Sigma)$, $\forall p$)

$$\bar{\Omega}_{ij} \stackrel{\text{by Def.}}{=} \frac{1}{2} \bar{R}_{ij,kl} \bar{\theta}^k \wedge \bar{\theta}^l \stackrel{\text{use } (*)}{=} K \bar{\theta}_i \wedge \bar{\theta}_j$$

Role of the signature of K :

Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called "de Sitter universes".

$K > 0$: $\Rightarrow \Sigma$ is a 3-dim. sphere (that can be embedded e.g. in a 4 dim euclidean (i.e. flat) space: closed universe

$K = 0$: $\Rightarrow \Sigma$ is euclidean $\mathbb{R}^3$. flat, infinite universe

$K < 0$: $\Rightarrow \Sigma$ is a 3-dim. "pseudo sphere". The constant negative curvature means it is everywhere "like a saddle". These Σ also possess ∞ volume.

□ Note: $\bar{R}$ and therefore K have units $\frac{1}{(\text{length})^2}$. Thus, by suitable choice of unit of length, we can choose units of length so that: (This is usually done in cosmology)

$$K = -1, 0 \text{ or } 1$$

A tetrad for spacetime: $g = -dt^2 + a^2(t)\bar{g}$

□ Define a convenient tetrad, i.e., ON basis of each $T_p(M)$,:

$$\theta^0 := dt$$

with t = cosmic time of above

$$\theta^i := a(t)\bar{\theta}^i$$

with $\bar{\theta}^i$ being the triad of Σ

□ Note: The $\bar{\theta}^i$ were chosen ON with respect to $\bar{g}$.

The θ^i are ON with respect to g .

We then have, e.g.:

* 1st structure equation on Σ : $\checkmark (i, j = 1, 2, 3)$

$$d\bar{\theta}^i + \bar{\omega}^i_j \wedge \bar{\theta}^j = 0 \quad (\Sigma 1)$$

Recall:

The Cartan structure equations express the torsion and curvature forms in terms of the connection form.
(2nd eqn: $\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$)

* 1st structure equation on M : $\checkmark (\mu, \nu = 0, 1, 2, 3)$

$$d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu = 0 \quad (M1)$$

Determine the 4-connection ω^μ_ν : (in spatially isotropic & homogeneous case)

Strategy: Calculate $d\theta^i$ in two ways:

$$\begin{aligned} 1.) \quad d\theta^i &= d(a\bar{\theta}^i) = (da) \wedge \bar{\theta}^i + a d\bar{\theta}^i \\ &= \left(\frac{da}{dt} dt\right) \wedge \bar{\theta}^i + a \underbrace{d\bar{\theta}^i}_{\text{use Eq. } \Sigma 1} \\ &= \dot{a} \theta^0 \wedge \bar{\theta}^i + a \bar{\omega}^i_j \wedge \bar{\theta}^j \end{aligned}$$

(use $a\bar{\theta}^i = \theta^i$) $\Rightarrow$

$$= \dot{a} \theta^0 \wedge \bar{\theta}^i + a \bar{\omega}^i_j \wedge \bar{\theta}^j$$

(use $\bar{\theta}^i = \frac{1}{a}\theta^i$
and $\theta^i \wedge \theta^0 = -\theta^0 \wedge \theta^i$) $\Rightarrow$

$$= -\frac{\dot{a}}{a} \theta^i \wedge \theta^0 + \bar{\omega}^i_j \wedge \theta^j \quad (A)$$

$$2.) \quad d\theta^i \stackrel{(M1)}{=} -\omega^i_\nu \wedge \theta^\nu = -\omega^i_0 \wedge \theta^0 - \omega^i_j \wedge \theta^j \quad (B)$$

Compare eqns A, B $\Rightarrow$

$$\omega^i_0 = \frac{\dot{a}}{a} \theta^i \quad \text{and} \quad \omega^i_j = \bar{\omega}^i_j$$

(Box)

(Intuition: expansion is nontrivial affine connection between space and time)

What is ω^0_0 ? Recall:

$$dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$$

But $dg_{\mu\nu} = 0$ for ON frames.

Thus $\omega_{\mu 0} = -\omega_{0\mu}$ here.

$$\Rightarrow \omega_{00} = 0$$

The curvature 2-form:

Recall: 2nd structure equations: (analogous to: $R_{\dots} = \Gamma + \Gamma + \Gamma\Gamma + \Gamma\Gamma$)

$$\Omega^\nu{}_\mu = d\omega^\nu{}_\mu + \omega^\nu{}_\lambda \wedge \omega^\lambda{}_\mu \quad (M2)$$

$$\bar{\Omega}^i{}_j = d\bar{\omega}^i{}_j + \bar{\omega}^i{}_k \wedge \bar{\omega}^k{}_j \quad (\Sigma2)$$

$\Rightarrow$ for $i, j \in \{1, 2, 3\}$ (afterwards we will calculate $\Omega^0{}_i, \Omega^i{}_0$)

$$\Omega^i{}_j \stackrel{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j \quad \text{use (Box)} \Rightarrow$$

$$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_k \wedge \bar{\omega}^k{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

$$\stackrel{\Sigma2}{=} \bar{\Omega}^i{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

Recall also:

$$\bar{\Omega}^i{}_j = K \bar{\theta}^i \wedge \bar{\theta}^j = \frac{K}{a^2} \theta^i \wedge \theta^j \quad (\text{It was a consequence of spatial isotropy \& homogeneity})$$

$$\Rightarrow \Omega^i{}_j = \frac{K}{a^2} \theta^i \wedge \theta^j + \frac{\dot{a}^2}{a^2} \theta^i \wedge \theta^j \leftarrow \begin{cases} \text{Recall from equations (box):} \\ \omega^0{}_i = \frac{\dot{a}}{a} \theta^i, \quad \omega_{0i} = -\frac{\dot{a}}{a} \theta^i \\ \omega_{i0} = \frac{\dot{a}}{a} \theta^i, \quad \omega^i{}_0 = \frac{\dot{a}}{a} \theta^i \end{cases}$$

$$\Rightarrow \boxed{\Omega^i{}_j = \frac{K + \dot{a}^2}{a^2} \theta^i \wedge \theta^j}$$

Similarly, one calculates: Exercise: check

$$\boxed{\Omega^0{}_i = \frac{\ddot{a}}{a} \theta^0 \wedge \theta^i}$$

Calculate the Einstein tensor:

Recall:

$$\Omega_{\mu\nu} = \frac{1}{2} R_{\mu\nu\sigma\epsilon} \theta^\sigma \wedge \theta^\epsilon$$

$\Rightarrow$ We can read off $R_{\mu\nu\sigma\epsilon}$.

⇒ We obtain the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R .

⇒ We obtain the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Result:

$$G_{00} = 3 \left(\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right)$$

$$G_{ii} = -2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}$$

$$G_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu$$

Exercise: verify

i.e., $G_{\mu\nu}$ is diagonal in this frame.

The energy-momentum tensor:

□ From $G_{\mu\nu} = 8\pi G T_{\mu\nu}$
we obtain that $T_{\mu\nu}$ must also be diagonal.

△ Recall the interpretation of the entries of a diagonal $T_{\mu\nu}$ in terms of matter energy density ρ , matter pressure p and cosmological constant Λ at the origin of geodesic coordinates:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} - \frac{1}{8\pi G} \Lambda \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(Why this factor here?
Because Λ was traditionally put on the LHS, with the curvature)

⇒ The only nontrivial dynamics of matter is here its equation of state:

$$\rho = \rho(p) \quad \text{or} \quad p = p(\rho) \quad !$$

What kind of matter causes such a $T_{\mu\nu}$?

Proposition:

The $T_{\mu\nu}$ of any F.L. spacetime is always of the form of that of a perfect fluid.

* The matter doesn't have to be fluid - it could also be e.g. a suitable quantum field.

* But the high symmetry of a F.L. spacetime requires that the matter's $T_{\mu\nu}$ matches that of a perfect fluid.

Proof:

Consider the 4-vector field dual to Θ^0 :

$$u = \frac{\partial}{\partial t} = e_0, \text{ i.e.: } u = u^\mu e_\mu \text{ with } u^0=1, u^i=0.$$

Using u , $T^{\mu\nu}$ takes the form that characterizes a perfect fluid:

$$T^{\mu\nu} = (s + p) \overset{\frac{\Lambda}{8\pi G}}{u^\mu u^\nu} + (p - \bar{\Lambda}) g^{\mu\nu}$$

Q: If the matter is a fluid, what's the vector field u ?

A: ^{i.e. our galaxy} We are a particle of the fluid and u is our velocity:

Why? u is tangent to timelike geodesics ^(that stand still in space because $u \perp e_i \forall i=1,2,3$)

Recall:

$$\begin{aligned} \omega^0_i &= \frac{1}{a} \theta^i \\ \omega_{0i} &= -\frac{a}{a} \theta^i \\ \omega^i_0 &= \frac{a}{a} \theta^i \\ \omega^i_j &= \frac{1}{a} \theta^i \\ \omega^0_0 &= 0 \end{aligned}$$

$$\nabla_u u = \nabla_{e_0} e_0 = \omega^{\mu}_0 \overset{\text{dual basis}}{(e_\mu)} = \frac{a}{a} \overset{=0}{\theta^i(e_\mu)} e_i = 0$$

The Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

now consists of:

$$G_{00} = 8\pi G T_{00} \Rightarrow$$

$$3 \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = 8\pi G \rho + \Lambda \quad \leftarrow \text{"Friedmann equation"} \quad (A)$$

$$G_{ii} = 8\pi G T_{ii} \Rightarrow$$

$$-2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi G p - \Lambda \quad (B)$$

□ Notice that Λ contributes

□ positively to the energy but

□ negatively to the pressure.

Observation: k/a^2 occurs in (A) and (B), i.e., we can eliminate it:

$-\frac{1}{2}a \left(\text{Eqn(B)} + \frac{1}{3} \text{Eqn(A)} \right)$ yields:

$$\ddot{a} = -\frac{1}{2}a 8\pi G \left(\frac{\rho}{3} + p \right) - \frac{1}{2}a \Lambda \left(-1 + \frac{1}{3} \right)$$

$\Rightarrow$

$$\ddot{a} = -\frac{4\pi G a}{3} (\rho + 3p) + \frac{1}{3}a \Lambda$$

Thus for all k : For ordinary matter must have deceleration, i.e., $\ddot{a} < 0$, but a positive cosm. constant Λ can make $\ddot{a} > 0$.

Experimental evidence?

- Supernova distance versus brightness data and evidence from cosmic background radiation:

$$\underline{\ddot{a} > 0 \text{ now!}}$$

⇒ At present, energy is already sufficiently diluted so that Λ dominates over ρ : $\approx 70\%$, Λ and $\approx 30\%$ ρ ^{matter} (dark + visible)

Note: p of a gas of galaxies is negligible.
Note: ρ includes dark matter.
Visible matter is only $\approx 3\%$.

- In the far future, ρ & p will have diluted $\rightarrow 0$, leaving only Λ . Then, the Friedmann eqn reads:

$$3 \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = \Lambda$$

Solutions:

$$a(t) = \begin{cases} \cosh(t\sqrt{\frac{\Lambda}{3}}) & \text{for } k=1 \\ \exp(t\sqrt{\frac{\Lambda}{3}}) & \text{for } k=0 \\ \sinh(t\sqrt{\frac{\Lambda}{3}}) & \text{for } k=-1 \end{cases}$$

Or something other than Λ will dominate T_{vac} then.
Experiments indicate that indeed a faster than exponential expansion may be under way. That cannot come from just Λ dominance alone.
See essay topic!

⇒ Exponential expansion is predicted!

General solution strategy with cosm. constant and matter:

- We have 3 unknown functions of time $a(t)$, $\rho(t)$, $p(t)$

and we have 3 equations that they obey:

Eqs. A, B and an equation of state $p = p(\rho)$ that depends on the "matter":

$$p_{\Lambda}(\rho) = -\rho_{\Lambda} \quad \text{for pure vacuum energy} \quad (\text{e.g., in very early universe})$$

$$p(\rho) = \frac{1}{3}\rho \quad \text{for pure radiation} \quad (\text{e.g., in the early universe})$$

$$p(\rho) = 0 \quad \text{for pure dust} \quad (\text{e.g., middle aged universe before } \Lambda \text{ took over})$$

- Observation:

(Eqn. A)

The Friedmann eqn. only contains a , ρ but not p !

Idea:

- this models the dilution of energy density
- Try to express ρ as a function of a to obtain: $\rho = \rho(a)$.
 - Using $\rho(a)$, the Friedmann eqn becomes an ordinary differential equation only for $a(t)$ and we are done!

Indeed, a key equation helps us to find $\rho(a)$:

Proposition: The Einstein eqns A, B, i.e., $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, imply:

$$\boxed{\frac{d}{da} (\rho a^3) = -3p a^2} \quad (P)$$

Indeed, when the parameter w in $p = w\rho$ is known, (P) yields $\rho(a)$:

- For dust, $p = 0 \Rightarrow \rho \sim a^{-3}$
 - For radiation, $p = \rho/3 \Rightarrow \rho \sim a^{-4}$
 - For pure Λ : $p = -\rho \Rightarrow \rho = \text{const}$
- ρ of radiation decays quicker than ρ of dust because radiation is not only diluted, its wavelengths are also stretched, which reduces the energy too.
- ρ of vacuum energy does not dilute!

Intuitive meaning of (P)?

- (P) is the GR version of the continuity equation for (i.e., without heat exchange with an environment) $\rightarrow$ adiabatic expansion: $dE = -p dV$
- With $V = a^3$, $E = \rho V$ it yields:
$$d(a^3 \rho) = -p d(a^3) = -3p a^2 da$$
- Thus: $\frac{d}{da} (a^3 \rho) = -3p a^2$ which is indeed (P).

Exact proof of proposition (P):

□ The Einstein equation $G^{\mu\nu} = 8\pi G T^{\mu\nu}$ and $G^{\mu\nu}_{;\nu} = 0$ imply $T^{\mu\nu}_{;\nu} = 0$

□ Here: $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}$

□ Thus:

$$0 = T^{\mu\nu}_{;\nu} = \overbrace{(\rho_{;\nu} + p_{;\nu})u^\mu u^\nu}^{\text{Leibniz rule}} + (\rho + p)u^\mu u^\nu_{;\nu} + p_{;\nu}g^{\mu\nu} \quad (g^{\mu\nu}_{;\nu} = 0)$$

(using $\nabla_{\mu} u^\mu = \nabla_\mu u^\mu \Rightarrow u^\mu \rho_{;\mu} = \nabla_\mu \rho$) $\Rightarrow$

$$= (\nabla_\mu \rho + \nabla_\mu p)u^\mu + (\rho + p)u^\mu \nabla_\mu u + p^{;\mu} \quad | \cdot u_\mu$$

(using $u^\mu u_\mu = -1$) $\Rightarrow$

$$= -\nabla_\mu \rho - \cancel{\nabla_\mu p} - (\rho + p) \nabla_\mu u + \underbrace{u_\mu p^{;\mu}}_{\cancel{\nabla_\mu p}}$$

$$\Rightarrow 0 = \nabla_\mu \rho + (\rho + p)(\nabla \cdot u) \quad (X)$$

It remains now to calculate $\nabla \cdot u$:

$$\nabla \cdot u = u^\lambda_{;\lambda} = \theta^\lambda(\nabla_{e_\lambda} e_0)$$

Recall: $\nabla_{e_b} e_c = \omega^c_b(e_b) e_c$

i.e.: $\nabla_{e_a} e_b = \omega^c_b(e_a) e_c \Rightarrow$

$$= \theta^\lambda(\overbrace{\omega^0_\lambda(e_\lambda)}^{\text{numbers}} e_0) = \omega^0_\lambda(e_\lambda) = \omega^i_0(e_i)$$

Recall that $\omega^i_0 = \frac{\dot{a}}{a} \theta^i \Rightarrow$

$$= \frac{\dot{a}}{a} \theta^i(e_i) = 3 \frac{\dot{a}}{a}$$

$\uparrow$ and $\theta^i(e_i) = \delta^i_i$

$\uparrow$ since $\omega^0_0 = 0$

$\Rightarrow$ Eqn. (X) becomes:

$$\nabla_\mu \rho + 3 \frac{\dot{a}}{a} (\rho + p) = 0 \quad (\text{Recall: } u = \frac{d}{dt})$$

Thus:

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = 0$$

$$\frac{d\rho}{dt} \frac{dt}{da} a^3 + 3\rho a^2 = -3pa^2$$

$$\frac{d\rho}{da} a^3 + 3\rho a^2 = -3pa^2$$

$\Rightarrow$

$$\frac{d}{da} (\rho a^3) = -3pa^2$$

this is Eqn (P) ✓