Note Title

More on singularity theorems

- Assume a set of symmetries of matter and spacetime has been chosen.
- Assume an exact solution or at least its asymptotic properties at early times have been found.
- Assume, we choose a timelike congruence e.g. of geodesics.

⇒ We can now explicitly calculate the twist, shear and expansion along the congruence:

The Hubble functions:

In particular, we can see how the expansion or contraction of the universe behaves dynamically e.g. when the condition of perfect isotropy is relaxed:

- Now we have different expansions in different directions, nonlinearly influencing another.
- Recall:

  The expansion in one direction can be enhanced by shear, as long as shear shrinks other directions.
- Definition:

  We define a rate of expansion tensor that includes shear:

$$\Theta_{\mu\nu} := \underbrace{\sigma_{\mu\nu}}_{\text{shear}} + \frac{1}{3}\,\theta\, h_{\mu\nu}$$

shear

projector $\perp$ to the timelike u-field

symmetric part of $B_{\mu\nu}$

expansion scalar.

- $\Theta_{\mu\nu}$ is fully spacelike and symmetric $\Rightarrow$ $\Theta_{\mu\nu}$ can be diagonalized in suitable ON frame $\{e_0, e_1, e_2, e_3\}$:

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & & & 0 \\ & \theta_1 & & \\ & & \theta_2 & \\ 0 & & & \theta_3 \end{pmatrix}$$

3 space-like directions.

with the traditional expansion being the trace (because $\sigma_{\mu\nu}$ is traceless):

$$\theta = \theta_1 + \theta_2 + \theta_3$$

why $\frac{1}{3}$? Recall that $Tr(h_{\mu\nu}) = 3$ $\Rightarrow$ is not quite projector

- **Definition:**

$H_i := \frac{1}{3}\theta_i$ — Local Hubble expansion function in direction e_i.

$H := \frac{1}{3}\theta$ — Overall local Hubble expansion function.

- **Definition:**

We use H_i, H to define local directional and general scale factors ℓ_i, ℓ:

The ℓ_i, ℓ are defined as the solutions to:

$$\frac{\dot{\ell}_i}{\ell_i} = H_i$$

$$\frac{\dot{\ell}}{\ell} = H$$

Here, the time derivative is defined as:

recall: u is timelike.

$$\dot{\ell} = u(\ell) = u^\mu \frac{\partial}{\partial x^\mu}\ell$$

- What behavior can occur in the far past?

Full set of cases not yet known.

But:

Explicit examples are known where e.g:

- All $\ell_i \to 0$ as in FL cosmologies
- $\ell_1, \ell_2 \to 0$, $\ell_3 \to \infty$ "cigar singularity"
- $\ell_1, \ell_2 \to 0$, $\ell_3 \to \text{const}$ "barrel singularity"
- $\ell_1, \ell_2 \to \text{const}$, $\ell_3 \to 0$ "pancake singularity"

Note: For homogeneous, isotropic FL models, H is the regular Hubble parameter and ℓ is its scale factor.

Singularity theorems for black holes

In 1915, Schwarzschild found this black hole solution to the Einstein equation:

Mass of black hole

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + d\varphi^2 \sin^2\theta\right)$$

Singularity: $r = 0$

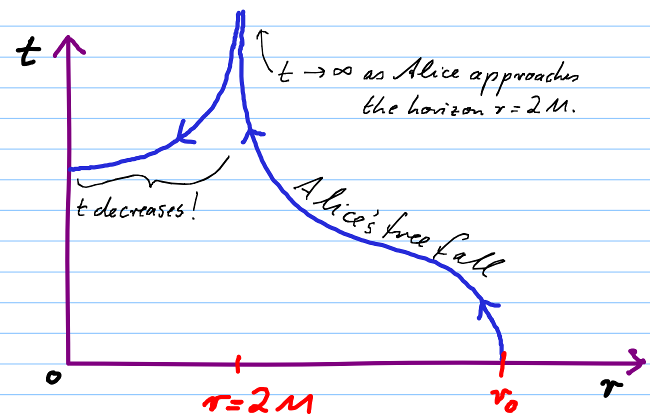
Horizon: $r = 2M$

Here, $x = (t, r, \varphi, \theta)$ are called the Schwarzschild coordinates.

Schwarzschild coordinates long misled intuition:

- The singularity at $r = 2M$ is not real: it disappears in other coordinate systems. The curvature is smooth across $r = 2M$.

- Due to the sign changes across $r = 2M$, for $r < 2M$ dt is spacelike and dr is timelike for $r < 2M$.
- Consider, for example, a traveler, Alice, who is freely falling from $r = r_0$ to $r = 0$:



$$r(\alpha) = \frac{r_0}{2}(1 + \cos(\alpha))$$

$$t(\alpha) = \left(\frac{r_0}{2} + 2M\right) w\alpha + \frac{r_0}{2} w \sin(\alpha) + 2M \log\left|\frac{w + \tan(\alpha/2)}{w - \tan(\alpha/2)}\right|$$

$$\tau(\alpha) = \frac{r_0}{2}\left(\frac{r_0}{2M}\right)^{1/2}(\alpha + \sin(\alpha))$$

Here: $0 < \alpha < \pi$ and $w := \left(\frac{r_0}{2M} - 1\right)^{1/2}$

$\Rightarrow$ need better choices of coordinate systems!

Simplification:

For now, we drop the φ and θ coordinates.

First design of a new cds (T, R) - Alice's choice (for $r_0 = 2M$):

- Require $g_{\mu\nu}(T,R)$ to be regular across $r = 2M$.
- Require $g_{\mu\nu}(0,0) = \eta_{\mu\nu}$ at $r = 2M$. If there's really no singularity at $r = 2M$ this must be possible.
- Extend this cds so that $g_{\mu\nu}(T,R) = f(T,R)\, \eta_{\mu\nu}$

⇒ Alice's choice are the Kruskal-Szekeres coordinates (T, R):

$$T(t,r) := 4M \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r-2M}{4M}} \left(\sinh\left(\frac{t}{4M}\right)\theta(r-2M) + \cosh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

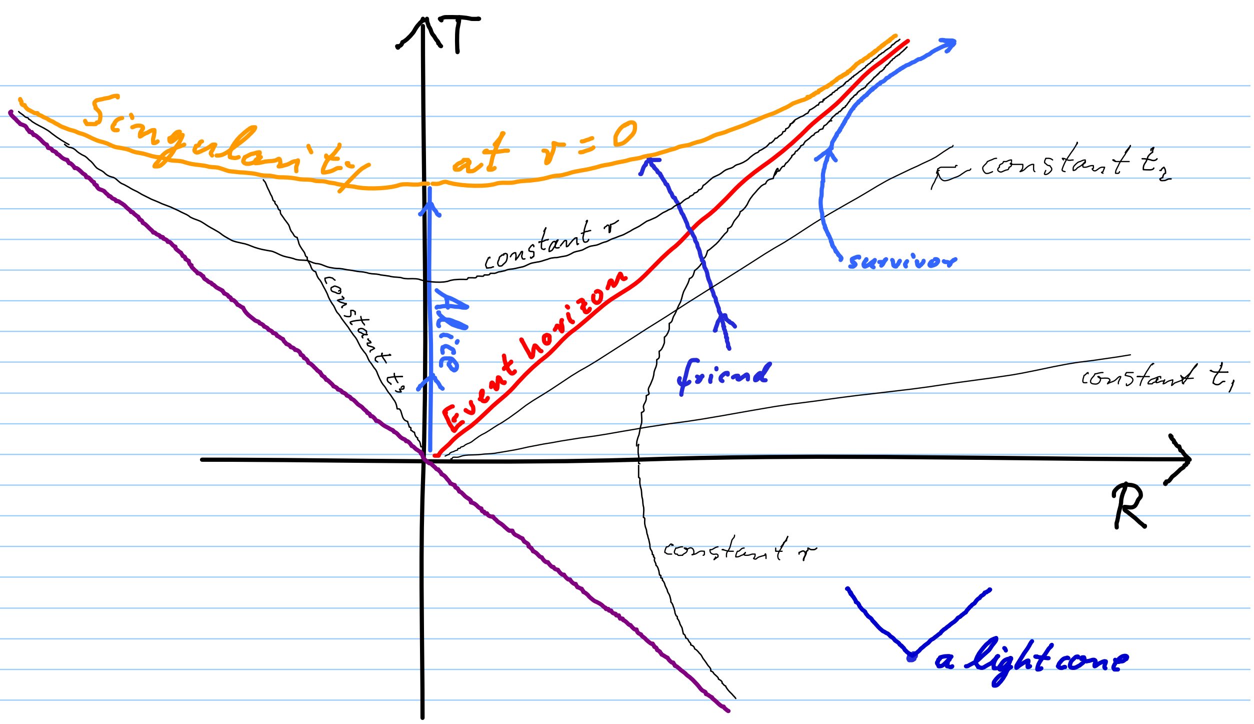
$$R(t,r) := 4M \left|\frac{r}{2M} - 1\right|^{1/2} e^{\frac{r-2M}{4M}} \left(\cosh\left(\frac{t}{4M}\right)\theta(r-2M) + \sinh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

This map is, in principle, invertible, to obtain $t(T,R)$, $r(T,R)$.

The Schwarzschild metric now takes this form:

$$ds^2 = \underbrace{\frac{2M}{r(T,R)} e^{1 - \frac{r(T,R)}{2M}}}_{\text{Conformal prefactor} = 1 \text{ as } r = 2M} \underbrace{\left(dT^2 - dR^2\right)}_{= \eta_{\mu\nu}}$$

Obeys all conditions!



- Alice was at rest at the event horizon.
- The singularity is at $T(R) = \left(R^2 + \frac{16M^2}{e}\right)^{1/2}$ and is spacelike.

Alice's light cone coordinates:

$$u := T - R, \quad v := T + R$$

Metric: $$ds^2 = \underbrace{\frac{2M}{r(u,v)}\, e^{1-\frac{r(u,v)}{2M}}}_{\substack{\text{conformal factor}\\ \text{(which is 1 at horizon)}}} \underbrace{du\, dv}_{\substack{\text{lightcone}\\ \text{Minkowski}}}$$

$\Rightarrow$ The action $S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\sqrt{g}\, d^2x$ becomes:

$$= \frac{1}{2}\int_{T>-R} \left(\partial_T\phi(T,R)\right)^2 - \left(\partial_R\phi(T,R)\right)^2 dT\, dR$$

$$= 2\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\partial_u\phi(u,v)\right)\left(\partial_v\phi(u,v)\right) dv\, du$$

$\leftarrow$ b/c region $T>-R$ means $T+R>0$, i.e. $v>0$.

$\Rightarrow$ Eqn of motion: $\partial_u\partial_v\phi(u,v) = 0$

Bob's choice of coordinate system

Bob is far from the black hole.

He wants a cds in which:

- $g_{\mu\nu}(x) \rightarrow \eta_{\mu\nu}$ as $r \rightarrow \infty$.
- $g_{\mu\nu}(x) = f(x)\,\eta_{\mu\nu}$ everywhere.

This is so that in his cds too

- photons travel at $45°$
- equations of motion of matter fields will be simple (useful in QFT!)

$\leadsto$ Bob's choice is the Tortoise coordinate system.

Tortoise cds (t^*, r^*):

- In terms of the Schwarzschild cds:

$$t^* := t$$

must require $r > 2M$!

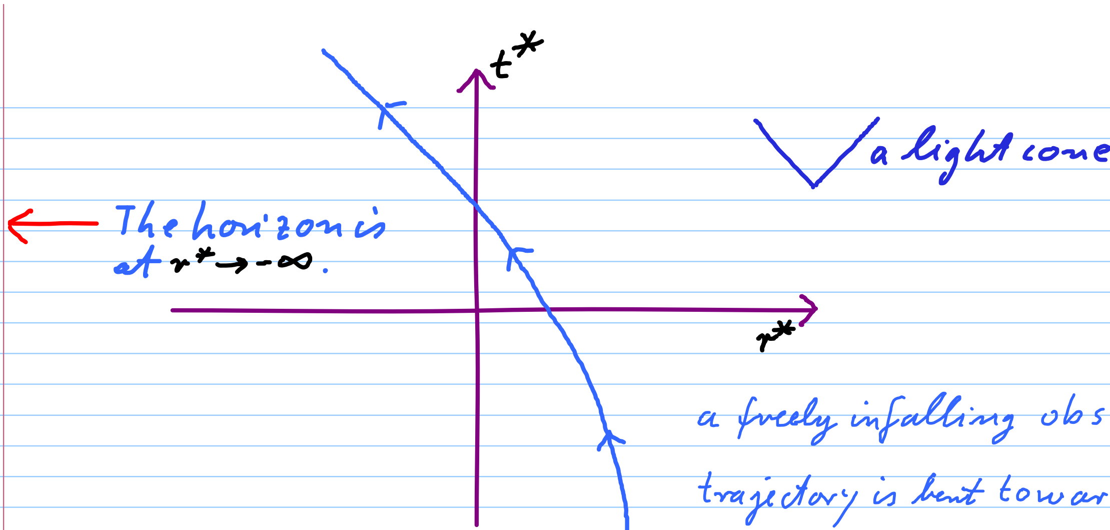
$$r^* := r - 2M + 2M \log\left(\frac{r}{2M} - 1\right)$$

$\Rightarrow$ Important: This is in principle invertible, to obtain $r = r(r^*)$ but only for $r > 2M$!

$\Rightarrow$ The tortoise cds only cover the BH's outside!

Metric: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(r^*)}\right)}\left(dt^{*2} - dr^{*2}\right)$$

Conformal factor $\to 1$ as $r \to \infty$, as planned but $\to 0$ at horizon.



a freely infalling observer's trajectory is bent towards 45° as he speeds up.

Bob's light cone coordinates: $\bar{u} := t^* - r^*$, $\quad \bar{v} := t^* + r^*$

The metric is then: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(\bar{u}, \bar{v})}\right)}_{\to 1 \text{ as } r \to \infty \text{ and } \to 0 \text{ as } r \to 2M} d\bar{u}\, d\bar{v}$$

⇒ The action:

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\,d^2x$$ becomes:

$$= \frac{1}{2}\int_{\mathbb{R}^2}\left(\partial_{t^*}\phi(t^*,r^*)\right)^2 - \left(\partial_{r^*}\phi(t^*,r^*)\right)^2 dt^* dr^*$$

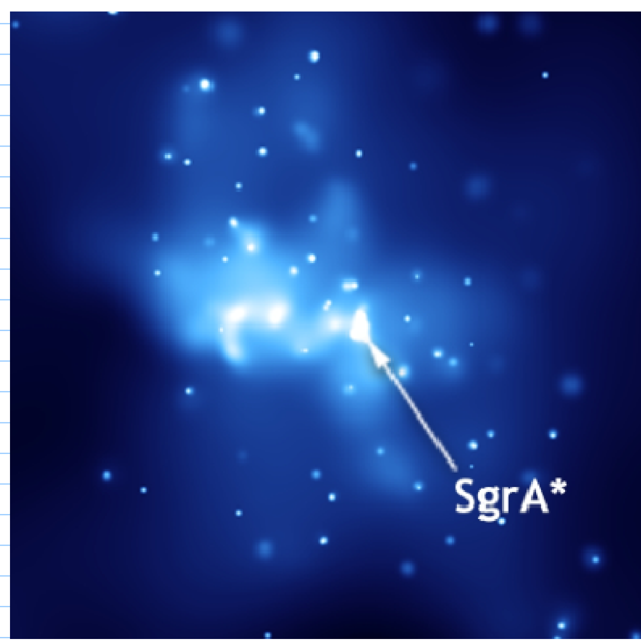
$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\partial_{\bar{u}}\phi(\bar{u},\bar{v})\right)\left(\partial_{\bar{v}}\phi(\bar{u},\bar{v})\right) d\bar{v}\, d\bar{u}$$

⇒ Eqn of motion: $\partial_{\bar{u}}\partial_{\bar{v}}\phi(\bar{u},\bar{v}) = 0$

Do real black holes possess a singularity?

Sagittarius A*

- 4 Mio stellar masses
- Diameter 44 Mio km
- 26000 lightyears away at centre of Milky Way.



→ Observations coming up 2018 by Event Horizon Telescope (in mm band) with hopefully enough resolution to see the event horizon.

How to model properties of real black holes roughly?

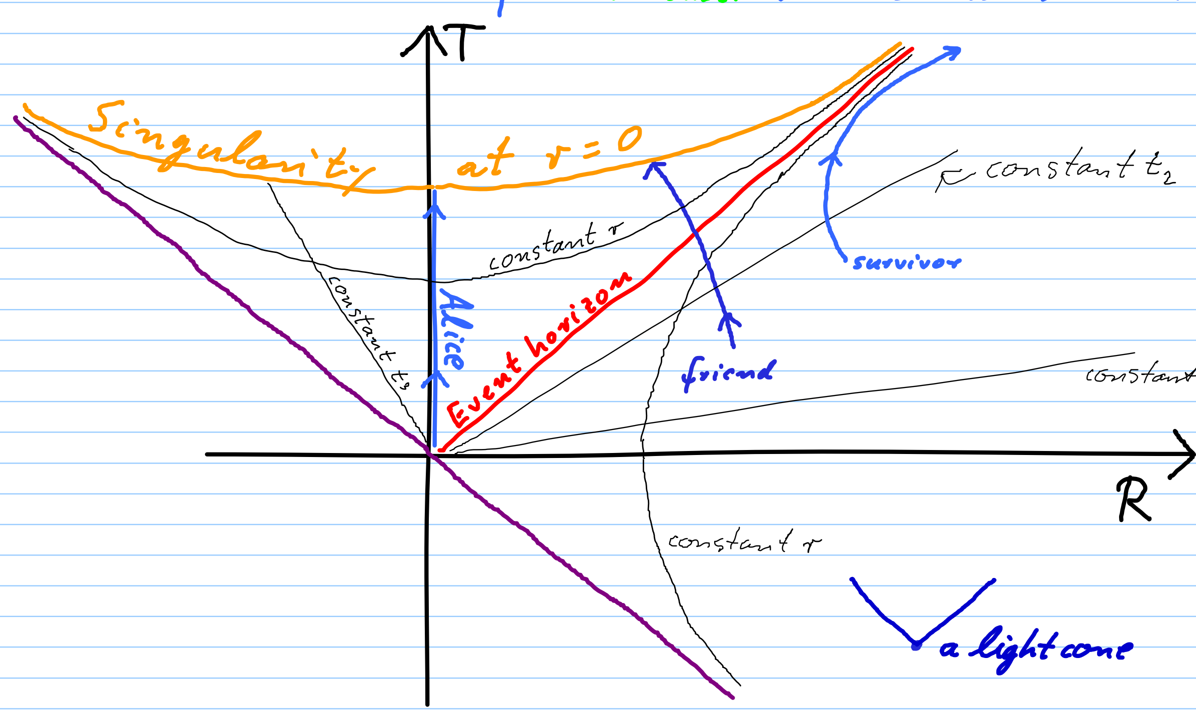
Singularity theorems suitable for black holes involve the concept and assumption of a trapped surface:

Def:

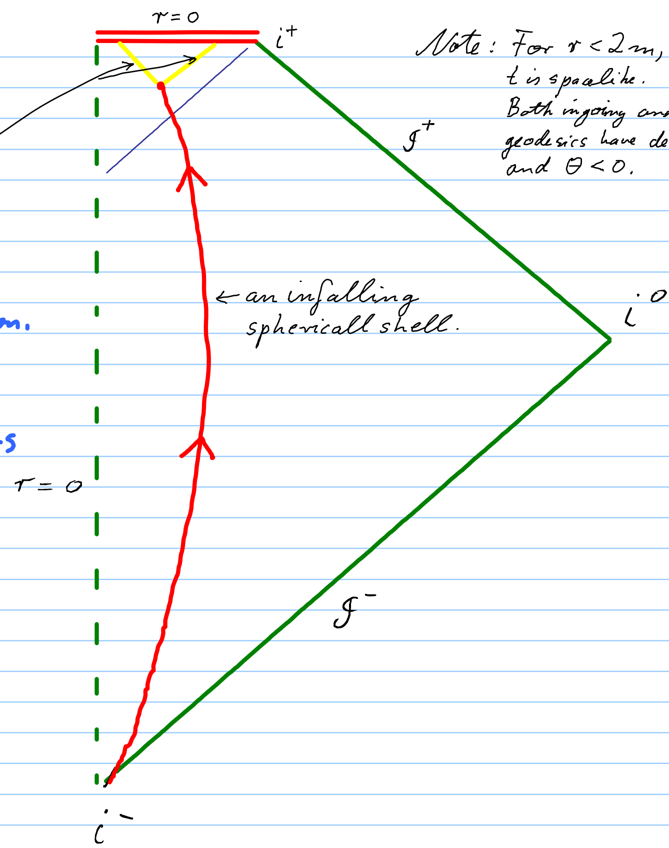
- Let Σ be a spacelike hypersurface. (Note: Σ is 3-dimensional)
- Let $T \subset \Sigma$ be a compact, 2-dimensional smooth spacelike submanifold of Σ. Consider the ingoing and the outgoing future-directed null geodesics that are orthogonal to T.
- If all these geodesics possess negative expansion, $\theta < 0$, then T is called a trapped surface.

Examples of trapped surfaces:

All spheres $r = \text{const.}$ inside a Schwarzschild black hole.



Generally:



Note: For $r<2m$, r is timelike and t is spacelike.
Both ingoing and outgoing null geodesics have decreasing r, and $\theta < 0$.

The in- and outgoing null geodesics both have negative expansion. Can't see it here b/c the neighboring geodesics are neighbors in the suppressed angular directions.

Def: Let Σ be a spacelike hypersurface.

Then, the (3-dim. spacelike) union, $\mathfrak{T}$, of all trapped surfaces $T \subset \Sigma$ is called the **trapped region** of Σ.

Def: The boundary $\partial\mathfrak{T} \subset \Sigma$ is called the **apparent horizon** of the spacelike hypersurface Σ.

Note: $\partial\mathfrak{T}$ is 2-dimensional and spacelike.

Def: If we foliate spacetime into spacelike hypersurfaces

$$\Sigma_\alpha,\ \alpha \in I \subset \mathbb{R}$$

each with its apparent horizon, $\mathfrak{T}_\alpha$, then their union

$$\mathcal{A} := \bigcup_\alpha \mathfrak{T}_\alpha$$

is called the **Trapping horizon** of the spacetime.

Remarks:

- To check for the existence of an event horizon
  $\mathcal{J}^-$ (worldline to i^+)
  in principle requires knowledge of the full future.

- But one can check for the existence of an apparent horizon in any spacelike hypersurface by calculating the expansions only at that time!

- The notion of apparent horizons is dependent on the the choice of foliation of spacetime into spacelike hypersurfaces.

- For static Schwarzschild black holes the event and apparent horizons coincide.

- Singularity theorems for black holes make assumptions that apparent horizons

Comment:

Hawking radiation is usually thought to emanate from the apparent horizon.