

The "physicist's definition of $T_p(M)$ "

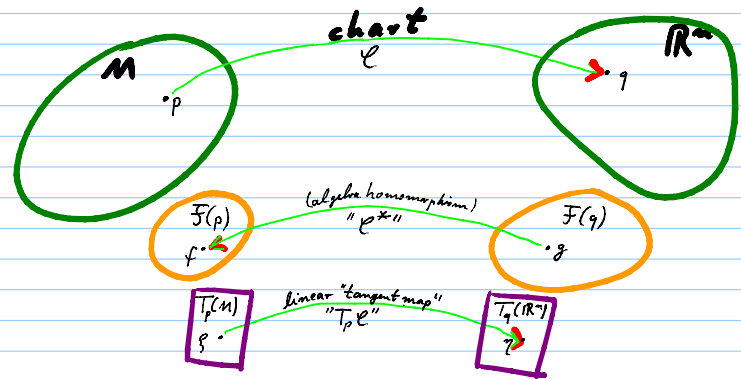
Recall: We obtain concrete representations for $p \in M$ and $f \in \mathcal{F}(p)$ and $\xi \in T_p(M)$ using a chart $\mathcal{C}: M \rightarrow \mathbb{R}^n$:

Recall: Def's used

pre-composition:

$$\mathcal{C}^*[\mathcal{F}] = \mathcal{F} \circ \mathcal{C}$$

$$T_p \mathcal{C}[\xi] = \xi \circ \mathcal{C}^*$$



Terminology: $\mathcal{C}^*$ is called the "pullback" of $\mathcal{C}$
 $T_p \mathcal{C}$ is called the "pullback" of $\mathcal{C}^*$

Namely:

- Each $p \in M$ has now a concrete image $q \in \mathbb{R}^n$, i.e., it has 'coordinates'.
- Each $f \in \mathcal{F}(p)$ is the image of a concrete function germ $g \in \mathcal{F}(q)$.

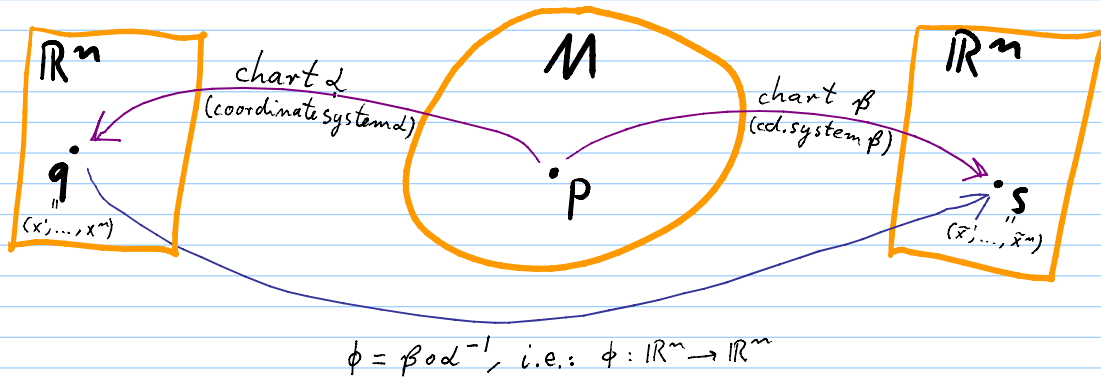
□ Each $\xi \in T_p(M)$ has now a concrete image $\eta \in T_q(\mathbb{R}^n)$

which we know has the form:

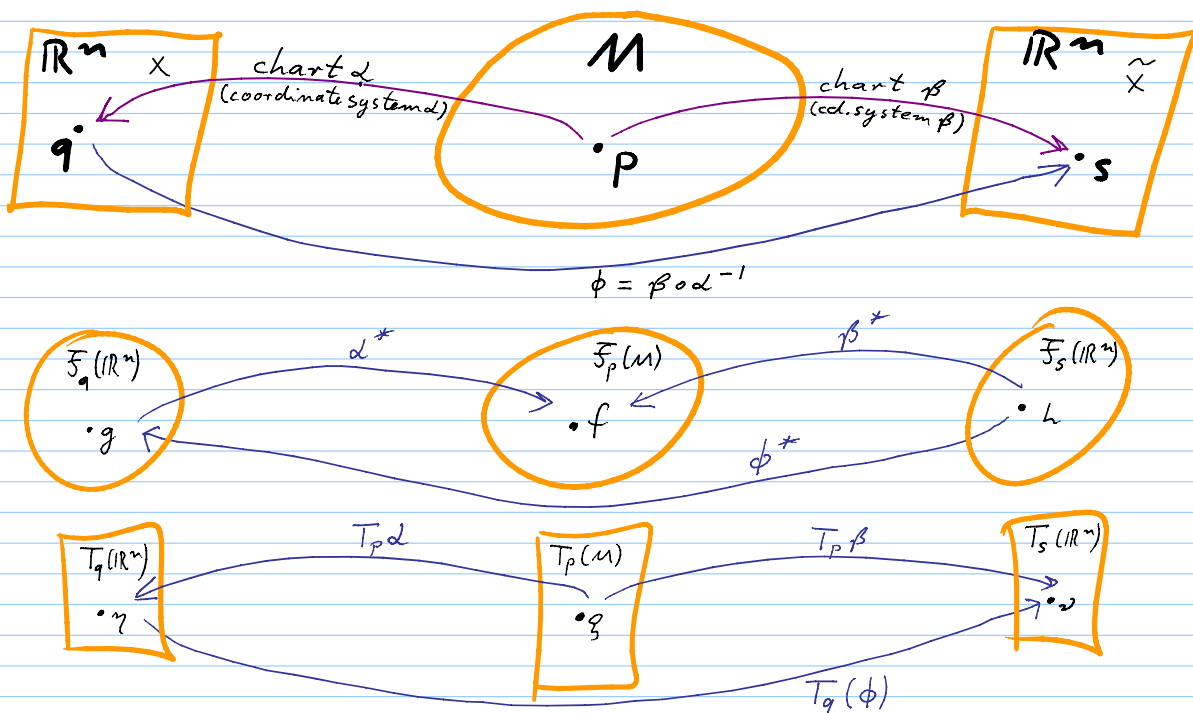
$$\eta = \sum_{i=1}^n \eta_i \frac{\partial}{\partial x_i} \Big|_{x=q} \quad \text{coefficients } \in \mathbb{R}$$

Question:

Given a $p \in M$ and a $\xi \in T_p(M)$,
how do their coordinates and coefficients
change under a change of charts?



$\Rightarrow$ When changing from chart α to chart β :



1. Every point $p \in \mathcal{M}$ now has 2 images,
 $q = (x^1, \dots, x^m)$ and $s = (\tilde{x}^1, \dots, \tilde{x}^m)$

$$(\tilde{x}^1, \dots, \tilde{x}^m) = \phi(x^1, \dots, x^m)$$

(concretely: $\tilde{x}^i = \phi^i(x^1, \dots, x^m)$).

2. Every function germ $f \in \mathcal{F}_p(\mathcal{M})$ has 2 pre-images,

$g \in \mathcal{F}_q(\mathbb{R}^m)$ and $h \in \mathcal{F}_s(\mathbb{R}^m)$, related by

$$f(p) = g(q) = h(s) \quad (\in \mathbb{R}) \quad \text{and by}$$

$$h(\tilde{x}^1, \dots, \tilde{x}^m) = g(x^1, \dots, x^m) \quad (*) \quad (\text{in a neighborhood})$$

3. Every tangent vector $\xi \in T_p(\mathcal{M})$ now has 2 images,
 $\eta \in T_q(\mathbb{R}^m)$ and $v \in T_s(\mathbb{R}^m)$.

By construction: (b/c of precomposition)

$$\eta(g) = \xi(f) = v(h) \quad (\in \mathbb{R})$$

$\Rightarrow$ in particular:

$$\sum_{i=1}^m \overbrace{\eta^i \frac{\partial}{\partial x^i}}^{\eta(g)} g(x^1, \dots, x^m) \Big|_{x=q} = \sum_{j=1}^m \overbrace{v^j \frac{\partial}{\partial \tilde{x}^j}}^{v(h)} \underbrace{h(\tilde{x}^1, \dots, \tilde{x}^m)}_{\substack{= \\ g(x^1, \dots, x^m) \text{ by } (*)}} \Big|_{\tilde{x}=s}$$

$$= \sum_{\substack{j=1 \\ k=1}}^m v^j \frac{\partial x^k}{\partial \tilde{x}^j} \Big|_{\tilde{x}=s} \frac{\partial}{\partial x^k} g(x^1, \dots, x^m) \Big|_{x=q}$$

Must be true for all q !

$$\Rightarrow \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} = \sum_{j=1}^n v^j \frac{\partial x^i}{\partial \tilde{x}^j} \bigg|_{\tilde{x}=s} \frac{\partial}{\partial x^i}$$

The $\{\frac{\partial}{\partial x^i}\}$ are linearly independent.

Jacobian matrix $D\phi^{-1}$
of ϕ^{-1} at s .

$\Rightarrow$

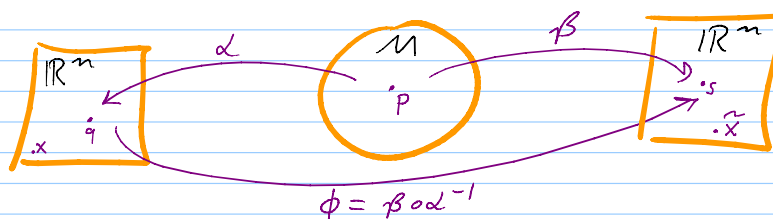
$$\eta^i = \sum_{j=1}^n \frac{\partial x^i}{\partial \tilde{x}^j} \bigg|_{\tilde{x}=s} v^j$$

Jacobian matrix $D\phi$
of ϕ at q .

$\Rightarrow$ conversely:

$$v^i = \sum_{j=1}^n \frac{\partial \tilde{x}^i}{\partial x^j} \bigg|_{x=q} \eta^j$$

Summary:



Given $\xi \in T_p(M)$, its images in charts α, β ,

namely $\eta = \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i}$ and $v = \sum_{i=1}^n v^i \frac{\partial}{\partial \tilde{x}^i}$, are

related by

Jacobian matrix $D\phi$

$$v^i = \sum_{j=1}^n \frac{\partial \tilde{x}^i}{\partial x^j} \bigg|_{x=q} \eta^j = \sum_{j=1}^n \frac{\partial \phi^i(x^1, \dots, x^n)}{\partial x^j} \bigg|_{x=q} \eta^j$$

This transformation property can also be used as the starting point for a definition of tangent vectors!

with: $\tilde{x}^i = \phi^i(x^1, \dots, x^n)$

→ The "physicist's definition of $T_p(M)$ "

Def: A tangent vector $\xi \in T_p(M)$ is a map that assigns to each (germ of a) chart a coefficient vector $\in \mathbb{R}^n$, so that if

□ $(\eta^1, \dots, \eta^n)$ is coefficient vector w. resp. to chart α

□ $(v^1, \dots, v^n)$ is coefficient vector w. resp. to chart β

then:

$$v^i = \sum_{j=1}^n \left. \frac{\partial \tilde{x}^i}{\partial x^j} \right|_{x=\beta(p)} \eta^j \quad \text{with } \tilde{x}^i = \phi^i(x) \\ \phi = \beta \circ \alpha^{-1}$$

So far, 2 equiv. defs. of $T_p(M)$:

In a chart, α , a tangent vector, $\xi \in T_p(M)$ is:

◦ algebraically: $\sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} \Big|_{x=\alpha(p)}$

i.e. it is a directional derivative

Defining property: Leibniz rule.

◦ physically: $(\eta^1, \dots, \eta^n)$

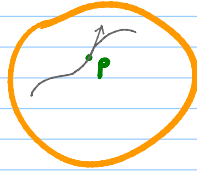
i.e. it is just the direction vector,

Defining property: chart change transformation rule

Finally:

The "geometric definition of $T_p(M)$ ":

Idea: Tangent vectors as tangents to paths.



Consider paths in M that pass through p :

$$\gamma: \mathbb{R} \rightarrow M$$

$$\gamma(0) = p$$

Note: For any $f: M \rightarrow \mathbb{R}$, we obtain:

$$f \circ \gamma: \mathbb{R} \rightarrow \mathbb{R}$$

Define:

Two diffable paths, γ_a, γ_b are called equivalent, if for all $f \in F_p(M)$:

$$\left. \frac{d}{dt} (f \circ \gamma_a) \right|_{t=0} = \left. \frac{d}{dt} (f \circ \gamma_b) \right|_{t=0} \quad (\otimes)$$

Intuition: Two paths γ_a, γ_b are equivalent if they have the same 'velocity' at p .

↑ Note: this includes speed and direction because $\otimes$ must hold for all $f \in F_p(M)$.

Definition: $T_p(M)^{(geom)}$ is the set of equivalence classes of diffable paths through p .

Are $T_p(M)^{(\text{geom})}$ and $T_p(M)^{(\text{alg})}$ equivalent?
we'll usually mean $T_p^{(\text{alg})}(M)$ when we write $T_p(M)$.

Yes!

really: each equivalence class of diffable paths through p
 Each path γ defines a linear map $\bar{\gamma}$:

$$\bar{\gamma}: T_p(M) \rightarrow \mathbb{R}$$

$$\bar{\gamma}: f \rightarrow \left. \frac{d}{dt} (f \circ \gamma) \right|_{t=0}$$

These $\bar{\gamma}$ obey the Leibniz rule:

$$\begin{aligned} \bar{\gamma}(fg) &= \left. \frac{d}{dt} (f \cdot g)(\gamma(t)) \right|_{t=0} = \left. \frac{d}{dt} (f(\gamma(t)) g(\gamma(t))) \right|_{t=0} \\ &= \left. \frac{d}{dt} f(\gamma(t)) \right|_{t=0} \overbrace{g(\gamma(0))}^{=p} + f(\gamma(0)) \left. \frac{d}{dt} g(\gamma(t)) \right|_{t=0} \\ &= \bar{\gamma}(f) g + f \bar{\gamma}(g) \quad \checkmark \end{aligned}$$

$\Rightarrow \bar{\gamma}$ is an element of $T_p(M)^{(\text{alg})}$

The "Cotangent Space" $T_p(M)^*$:

Recall:

Given an n -dimensional vector space V , the set of linear maps $\omega: V \rightarrow \mathbb{R}$ forms also an n -dim. vector space. It is called the "dual space", and denoted V^* .

Definition:

The dual vector space to $T_p(M)$ is called the **Cotangent Space**, and denoted $T_p(M)^*$.

We notice:

For every (germ of a) function at p ,
 $f \in \mathcal{F}(p)$

one naturally obtains an element
" $df \in T_p(M)^*$ "

called the "differential of f ."

Namely:

$df : T_p(M) \rightarrow \mathbb{R}$ is the linear map:

$$df : \xi \rightarrow \xi(f)$$

(Note: thus, we can view " d " as a map: $d : \mathcal{F}_p(M) \rightarrow T_p(M)^*$. See later...)

Concretely: in a cds., i.e., in a chart,

the abstract $\xi \in T_p(M)$ and $f \in \mathcal{F}(p)$

correspond to some $\eta \in T_q(\mathbb{R}^n)$ and $g \in \mathcal{F}(q)$.

Then: $\overset{T_p(M)^*}{\downarrow} dg : T_q(\mathbb{R}^n) \rightarrow \mathbb{R}$

$$dg : \eta \rightarrow \eta(g) = \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} \bigg|_{x=q} g(x^1, \dots, x^n)$$

Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} \bigg|_{x=q}$

a basis of $T_q(\mathbb{R}^n)$ is $\left\{ \frac{\partial}{\partial x^i} \bigg|_{x=q} \right\}_{i=1}^n$

Question: What is the dual basis in $T_q(\mathbb{R}^n)^*$?

□ Consider the coordinate functions: $x^k: \mathbb{R}^n \rightarrow \mathbb{R}$.

□ Their differentials $dx^k \in T_q(\mathbb{R}^n)^*$ obey:

$$dx^k: T_q(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$dx^k: \left. \frac{\partial}{\partial x^i} \right|_{x=q} \rightarrow \left. \frac{\partial}{\partial x^i} x^k \right|_{x=q} = \delta_i^k$$

$\Rightarrow$ The dual basis in $T_q(\mathbb{R}^n)^*$ is given by

$$\{dx^k\}_{k=1}^n$$

Thus:

Every element $\omega \in T_q(\mathbb{R}^n)^*$ takes the form:

$$\omega = \sum_{i=1}^n \omega_i dx^i$$

$\nwarrow \in \mathbb{R}$

and its action is:

$$\omega: T_q(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$\begin{aligned} \omega: \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} &\rightarrow \sum_{i=1}^n \omega_i dx^i \left(\sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} \right) \\ &= \sum_{i=1}^n \omega_i \underbrace{\sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} x^i}_{= \delta_j^i} \\ &= \sum_{i=1}^n \omega_i \eta^i \end{aligned}$$

$$\Rightarrow \omega \left(\sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n \omega_i \eta^i \quad (\text{I})$$

In particular: For arbitrary $g \in \mathcal{F}(q)$, its differential $dg \in T_q(\mathbb{R}^n)^*$ must be of the form:

$$dg = \sum_{k=1}^n \omega_k dx^k \text{ with suitable } \omega_k \in \mathbb{R}.$$

↑ How to calculate them?

We know:

$$dg(\gamma) = \gamma(g) = \sum_{i=1}^n \gamma^i \underbrace{\frac{\partial}{\partial x^i} g(x)}_{\omega_i} \Big|_{x=q} \quad (\text{II})$$

Compare I, II $\Rightarrow \omega_i = \frac{\partial}{\partial x^i} g(x) \Big|_{x=q}$

$$\Rightarrow dg = \sum_{i=1}^n \left(\frac{\partial}{\partial x^i} g(x) \Big|_{x=q} \right) dx^i$$

Exercise: (the "pull back" map)

Assume that $\beta \in T_p(M)^*$, under two charts α, β , as above, corresponds to $\omega \in T_q(\mathbb{R}^n)^*$ and $\mu \in T_s(\mathbb{R}^n)^*$ with:

$$\omega = \sum_{i=1}^n \omega_i dx^i \text{ and } \mu = \sum_{i=1}^n \mu_i d\tilde{x}^i$$

Show that $\mu_i = \sum_{j=1}^n \frac{\partial x^j}{\partial \tilde{x}^i} \Big|_{\tilde{x}=s} \omega_j$

↑ Notice that this is the inverse of the Jacobian matrix of $\beta \circ \alpha^{-1}$ at q

Remark: The physicist's definition of $T_p(M)^*$ uses this.

Some notation and terminology:

- Elements of $T_p(M)$ are called contravariant vectors
- Elements of $T_p(M)^*$ are called covariant vectors
- One often writes symbolically

$$\xi = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} \Big|_p \quad \text{for } \xi \in T_p(M)$$

$$\omega = \sum_{i=1}^n \omega_i dx^i \quad \text{for } \omega \in T_p(M)^*$$

even without specifying a particular chart.

We are now ready to define tensors:

Def: A tensor, t , of rank (r, s) is an element of $T_p(M)_s^r := \underbrace{T_p(M) \otimes \dots \otimes T_p(M)}_{r \text{ factors}} \otimes \underbrace{T_p(M)^* \otimes \dots \otimes T_p(M)^*}_{s \text{ factors}}$

In a chart: $t = \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_s}} t_{j_1, \dots, j_s}^{i_1, \dots, i_r} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$
 $\uparrow$
 $\mathbb{R}$

Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$\bar{t}_{j_1, \dots, j_s}^{i_1, \dots, i_r} = \sum_{\substack{k_1, \dots, k_r \\ l_1, \dots, l_s}} \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \dots \frac{\partial \bar{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}} \dots \frac{\partial x^{l_s}}{\partial \bar{x}^{j_s}} t_{l_1, \dots, l_s}^{k_1, \dots, k_r}$$

Thus: $T_p(M) = T_p(M)^1$ and $T_p(M)^* = T_p(M)_1$, i.e.:

- a tangent vector is a tensor of rank $(1, 0)$
- a cotangent vector is a tensor of rank $(0, 1)$

Finally: From local to global!

Def: We call $T(M) := \bigcup_{p \in M} (p, T_p(M))$,
the Tangent bundle.
↑ a "base point"
↑ a "fibre"

Note: $T(M)$ is itself a manifold. It is $2n$ -dimensional.

Def: $T(M)$ is then also called the "Total Space".

Def: M is also called the "Base Space".

Recall that all $T_p(M)$ are n -dimensional real vector spaces, i.e., are isomorphic to $\mathbb{R}^n$.

Def: We therefore call $\mathbb{R}^n$ the "Standard Fibre".

Remark: One obtains other Fibre bundles by choosing other standard fibers.

E.g.: $\square$ Co-tangent bundle $T^*(M)$

$\square$ (r,s) -tensor bundle $T^*_{rs}(M)$

$\square$ Bundles for isospinors (vector bundles) and gauge groups (principle bundles)

Def: The map $\pi: T(M) \rightarrow M$
 $\pi: (p, T_p(M)) \rightarrow p$ (i.e.: $\pi^{-1}(p) = T_p(M)$)
is called the "Bundle Projection".

Def: A Section, σ , is a map, $\sigma: M \rightarrow T(M)$, which is a continuous right inverse of π :

$$\pi(\sigma(x)) = x \quad \forall x \in M \quad (\text{i.e.: } \pi \circ \sigma = \text{id})$$

Notice: The graph of a "field" is a section of its fibre bundle.

Recall: The graph of a function $f: A \rightarrow B$ is:

$$\{(a, f(a))\}_{a \in A}$$

Def: $\square$ A tangent vector field is a map $\xi: \overset{M}{\downarrow} p \rightarrow \overset{T_p(M)}{\downarrow} \xi_p$

In a chart: $\xi = \sum_{i=1}^n \xi^i(x) \frac{\partial}{\partial x^i}$

$\square$ A cotangent vector field is a map $\omega: \overset{M}{\downarrow} p \rightarrow \overset{T_p^*(M)}{\downarrow} \omega_p$

In a chart: $\omega = \sum_{i=1}^n \omega_i(x) dx^i$

$\square$ Similarly, tensor fields: $t: \overset{M}{\downarrow} p \rightarrow \overset{T_p(M)^{\otimes k}}{\downarrow} t_p$

In a chart: $t = \sum t_{i_1, \dots, i_k}(x) \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_k}}, dx^{i_1} \dots dx^{i_k}$

Why then fibre bundles? To capture global nontriviality.

So far, not a concern with GR, but it does come up with gauge theories.

Fibre bundles are required to be locally trivial:

M can be covered with neighborhoods U_α ,

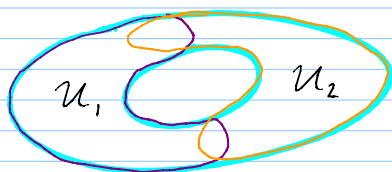
so that

means: there exists a differentiable isomorphism

or other standard fibre for other fibre bundles.

$$\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^n$$

But fibre bundles are allowed to be globally nontrivial:



For a suitable vector bundle B , we can have

$$\pi^{-1}(U_1) \cong U_1 \times \mathbb{R}^n$$

$$\pi^{-1}(U_2) \cong U_2 \times \mathbb{R}^n$$

but in the overlap regions, the two isomorphisms may differ $\Rightarrow B \neq M \times \mathbb{R}^n$

(The isomorphisms may differ by elements of $GL_n(\mathbb{R})$, the "structure group" here)

Definition: For the algebra of C^∞ functions $M \rightarrow \mathbb{R}$ we write $\mathcal{F}(M)$.

Note: One can show that contravariant vector fields are the derivations of the algebra $\mathcal{F}(M)$, i.e.:

If ξ is a contravariant vector field, then

$$\xi: \mathcal{F}(M) \rightarrow \mathcal{F}(M)$$

is linear and obeys the Leibniz rule:

$$\xi(fg) = \xi(f)g + f\xi(g)$$

for all $f, g \in \mathcal{F}(M)$.

Next topic: Differential forms:

We already have covered some differential forms:

- The set $\Lambda_0 := \mathcal{F}(M)$ is called the set of 0-forms.
- The set of covariant vector fields is denoted Λ_1 and called the set of 1-forms.
- For $r = 2, 3, \dots$ the set, Λ_r , of r -forms is defined to be the set of totally anti-symmetric tensor fields of rank $(0, r)$.