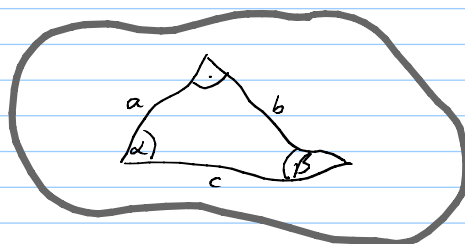


How to describe the "shape" of a manifold?

Historically:



E.g., on a potato-shaped surface:

$$a^2 + b^2 \neq c^2$$

$$\alpha + \beta + 90^\circ \neq 180^\circ$$

Helmholtz & Gauss already considered checking for curvature of space this way.

Recall:

Defined $g_{\mu\nu}(x)$

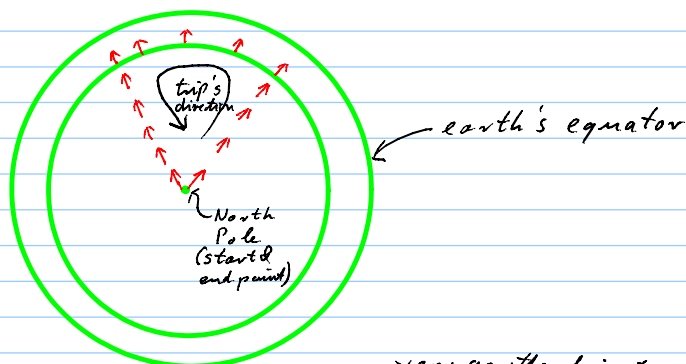
$\Rightarrow$ infinitesimal distances $\Rightarrow$ finite distances $\Rightarrow$ shape

Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the nontriviality of the parallel transport of vectors on the manifold:

Example:

- start with a vector at North Pole.
- parallel transport down to some lower latitude, along that latitude and then back to pole.
- vector will arrive at pole rotated, in spite of having only been parallel transported!



This motivates:

Try to define local shape through "derivative" of vectors with respect to parallel transport!

Recall: Lie derivative insensitive to g . Indeed, for $\xi_1 = \frac{\partial}{\partial x^1}$, $\xi_2 = \frac{\partial}{\partial x^2}$, we have $[L_{\xi_1}, L_{\xi_2}] = L_{[\xi_1, \xi_2]} = 0 \Rightarrow$ No shape info from L_{ξ} !

The Covariant Differentiation, ∇ :

Definition: Any linear map of tangent vector fields

$$\nabla : T'(M) \times T'(M) \rightarrow T'(M)$$

$$\nabla : \quad \eta, \xi \rightarrow \nabla_{\xi} \eta$$

obeying

$$(I) \quad \nabla_{f\xi} \eta = f \nabla_{\xi} \eta, \quad \forall f \in \mathcal{F}(M)$$

$$(II) \quad \nabla_{\xi}(f\eta) = \underbrace{\xi(f)}_{\nabla_{\xi} f} \eta + f \nabla_{\xi} \eta \quad (\text{Leibniz rule})$$

is called a covariant derivative or affine connection.

Note:

For now, let us assume a metric has not (yet) been specified, so we are free to choose ∇ , and this choice defines the shape of M !

∇ in a chart: $\square$ Choose as bases for $T_x(M)$, e.g.: $\left\{ \frac{\partial}{\partial x^i} \right\}$

$\square$ Given a covariant derivative ∇ , consider its action on basis vectors, such as, e.g.: $\xi = \frac{\partial}{\partial x^i}, \eta = \frac{\partial}{\partial x^j}$:

Recall: $L_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k}$$

The Γ^k_{ij} are called "Christoffel symbol" or "Connection coefficients".

Thus, via the axioms:

$$\begin{aligned} \nabla_{\xi} \eta &= \nabla_{\xi^i \frac{\partial}{\partial x^i}} \left(\eta^j \frac{\partial}{\partial x^j} \right) \stackrel{(I)}{=} \xi^i \nabla_{\frac{\partial}{\partial x^i}} \left(\eta^j \frac{\partial}{\partial x^j} \right) \\ &\stackrel{(II)}{=} \xi^i \left(\eta^j{}_{,i} \frac{\partial}{\partial x^i} + \eta^j \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k} \right) \\ &= (\xi^i \eta^k{}_{,i} + \xi^i \eta^j \Gamma^k_{ij}) \frac{\partial}{\partial x^k} \end{aligned}$$

function
vector

Notation:

$$\eta^k{}_{;i} := \eta^k{}_{,i} + \eta^j \Gamma^k_{ij}$$

↑ semi-colon for covariant derivatives

Thus:

$$\nabla_{\xi} \eta = \xi^i \eta^k{}_{;i} \frac{\partial}{\partial x^k} \quad (*)$$

Important: the Γ^k_{ij} transform non-tensorially when $x \rightarrow \bar{x}$:

On one hand: because $\frac{\partial}{\partial \bar{x}}$ is tangent vector

$$\nabla_{\frac{\partial}{\partial \bar{x}^a}} \frac{\partial}{\partial \bar{x}^b} = \bar{\Gamma}^c_{ab} \frac{\partial}{\partial \bar{x}^c} = \bar{\Gamma}^c_{ab} \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial}{\partial x^k} \quad (I)$$

On the other hand:

$$\begin{aligned} \nabla_{\frac{\partial}{\partial \bar{x}^a}} \frac{\partial}{\partial \bar{x}^b} &= \nabla_{\underbrace{\frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial}{\partial x^i}}_{\in F(M)}} \left(\frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j} \right) && \text{use axiom (b)} \Rightarrow \\ &= \frac{\partial x^i}{\partial \bar{x}^a} \nabla_{\frac{\partial}{\partial x^i}} \left(\underbrace{\frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j}}_{\in F(M)} \right) && \text{use Leibniz rule (c)} \Rightarrow \\ &= \frac{\partial x^i}{\partial \bar{x}^a} \left[\left(\frac{\partial}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right] \\ &= \left(\frac{\partial}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad (II) \end{aligned}$$

Compare I, II $\Rightarrow$

$$\bar{\Gamma}^c_{ab} \frac{\partial x^k}{\partial \bar{x}^c} = \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij} \quad \left(\frac{\partial \bar{x}^r}{\partial x^k} \Rightarrow \right)$$

$\Rightarrow$

$$\bar{\Gamma}^r_{ab} = \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b}}_{\text{This term is indep. of } \Gamma} + \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij}}_{\text{only this term would be there, if the } \Gamma^k_{ij} \text{ were tensor coefficients in the } \frac{\partial}{\partial x^i}, dx^i \text{ bases.}}$$

This term is indep. of Γ
 $\Rightarrow \Gamma$ can be zero in one coordinate system and nonzero in another!

only this term would be there, if the Γ^k_{ij} were tensor coefficients in the $\frac{\partial}{\partial x^i}, dx^i$ bases.

(Can be shown to be equivalent)



Physicists' definition of ∇ : Any set of n^3 functions $\Gamma^k_{ab}(x)$ which transform this way are defining a cov. derivative ∇ .

The "absolute" covariant derivative ∇ :

Consider the covariant derivative but:
without choosing a direction vector ξ :

$$\nabla : T_x(M)' \rightarrow T_x(M)',$$

$$\nabla : \eta = \eta^i_{(x)} \frac{\partial}{\partial x^i} \rightarrow \nabla \eta(x) = \eta^k_{;i}(x) dx^i \otimes \frac{\partial}{\partial x^k}$$

(I.e. feed the open covariant slot
of $\nabla \eta$ with contravariant ξ .)

Indeed: The contraction of $\nabla \eta$ with ξ yields:

$$\nabla \eta(\xi) = \eta^k_{;i} \underbrace{dx^i(\xi)}_{\substack{\uparrow \\ dx^i(\xi) = \xi(x^i) = \xi^j \frac{\partial}{\partial x^j} x^i = \xi^j \delta^i_j = \xi^i}} \frac{\partial}{\partial x^k} = \eta^k_{;i} \xi^i \frac{\partial}{\partial x^k} = \nabla_{\xi} \eta \quad \text{ok with } (*)$$

We defined ∇ algebraically. Now, extract the

Geometric meaning of ∇ :

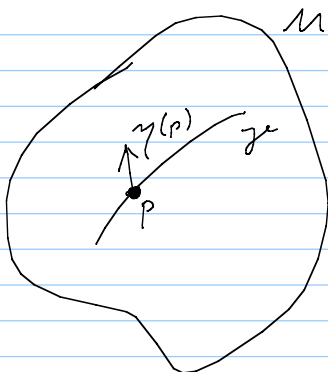
(∇ describes infinitesimal parallel transport
It should also describe finite parallel transport)

Definition: Assume ∇ is given. Choose a path $\gamma: \mathbb{R} \rightarrow M$.

Then, a tangent vector field η is
called auto-parallel along γ , if

$$\nabla_{\dot{\gamma}} \eta = 0$$

i.e. if η doesn't change under parallel
transport along the path γ .



Note: We'll see that they always exist, i.e., we
can always parallel transport a vector η finite distances.

□ In a chart,

$$\eta = \eta^i(x) \frac{\partial}{\partial x^i}$$

and

$$\gamma: [a, b] \rightarrow M$$

$$\gamma: t \rightarrow x^i(t)$$

and the tangent vector:

$$\dot{\gamma}(x(t)) = \frac{dx^k}{dt} \frac{\partial}{\partial x^k}$$

Thus: $\nabla_{\dot{\gamma}} \eta = \nabla_{\frac{dx^k}{dt} \frac{\partial}{\partial x^k}} \left(\eta^i \frac{\partial}{\partial x^i} \right) = \frac{dx^k}{dt} \nabla_{\frac{\partial}{\partial x^k}} \left(\eta^i \frac{\partial}{\partial x^i} \right)$

$$= \frac{dx^k}{dt} \left(\frac{\partial \eta^i}{\partial x^k} \frac{\partial}{\partial x^i} + \eta^i \Gamma^i_{ki} \frac{\partial}{\partial x^i} \right)$$
$$= \left(\frac{d\eta^i}{dt} + \frac{dx^k}{dt} \eta^j \Gamma^i_{kj} \right) \frac{\partial}{\partial x^i} \stackrel{!}{=} 0$$

$\Rightarrow$ η autoparallel to γ means:

$$\frac{d\eta^i}{dt} + \eta^j \frac{dx^k}{dt} \Gamma^i_{kj} = 0$$

i.e. this is the condition for the vectors of η being parallel translates of each other, along γ .

□ Conclusion:

This is 1st order ODEs for η . Thus:

Initial condition $\eta(\gamma(0)) \Rightarrow$ solution $\eta(\gamma(t))$ exists at least locally

$\Rightarrow$ □ Proposition:

Given a path $\gamma: [t, s] \rightarrow M$, the

autoparallel transport of a tangent vector η at $\gamma(t)$ to $\gamma(s)$ is unique.

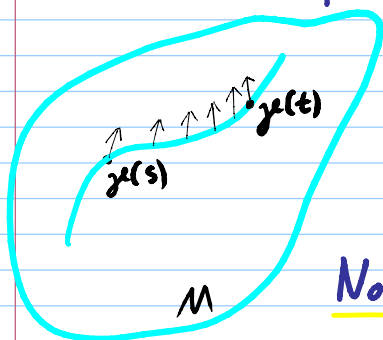
I.e., the path γ defines a parallel transport map τ :

$$\tau(t,s): T_{\gamma(s)} \rightarrow T_{\gamma(t)}$$

$$\tau(t,s): \eta(\gamma(s)) \rightarrow \eta(\gamma(t))$$

□ Q: Can one use τ to obtain ∇ as a Newton-Leibniz limit?

□ Proposition: (for the proof, see e.g. the text by Straumann)



$$\nabla_{\dot{\gamma}} \eta(\gamma(t)) = \left. \frac{d}{ds} \right|_{s=t} \tau(s,t) (\eta(\gamma(s)))$$

Note: Since we can choose paths with arbitrary $\dot{\gamma}$ this equation can be used as a geometric definition of ∇ .

∇ for arbitrary tensors:

□ The parallel transport map $\tau(s,t)$ transports tangent vectors η from $\gamma(s)$ to $\gamma(t)$.

□ Definition: $\tau(s,t)$ also parallel transports the dual vectors ω , namely so that contraction is conserved:

$$\underbrace{\tau(\omega)}_{\text{parallel transported } \omega} (\underbrace{\tau(\eta)}_{\text{parallel transported } \eta}) = \omega(\eta) \quad (C)$$

□ Extension of τ to tensor products:

$$\tau(S_1 \otimes S_2) := \tau(S_1) \otimes \tau(S_2) \quad (T)$$

↑ ↑
 S_1 and S_2 are tensors of arbitrary rank.



Definition:

arbitrary tensor $\nabla_{\xi} S'(p)$ $\swarrow$ arb. point $p \in M$

arb. tangent vector $\nearrow$

$$\nabla_{\xi} S'(p) := \nabla_{\dot{\gamma}} S'(\gamma(t)) \Big|_{t=0}$$

$$:= \frac{d}{dt} \Big|_{t=0} \tau(t, 0)(S'(\gamma(t)))$$

here, γ is any path through p obeying:

$$\dot{\gamma}(0) = \xi(p), \quad \gamma(0) = p$$

Exercise:

Show that when S is a scalar function $S \in F(M)$, then:

$$\nabla_{\xi} S = \xi(S) = \xi^i \frac{\partial}{\partial x^i} S$$

Absolute covariant derivative:

(for abs. derivative one is not specifying the direction.) $\nabla S'$ $\xrightarrow{\text{fed to } \nabla S \text{ which is } (r, p) \text{ tensor}}$

$$(\nabla S')(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_r, \xi) := \nabla_{\xi} S'(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_r)$$

Properties of ∇ :

* ∇ is a derivation:

(because ∇ inherits the Leibniz rule from $\frac{d}{ds}$)

$$\nabla_{\xi}(S_1 \otimes S_2) = \frac{d}{ds} \Big|_{s=t} \tau(S_1 \otimes S_2) = \frac{d}{ds} \Big|_{s=t} \tau(S_1) \otimes \tau(S_2)$$

$$= \left[\frac{d}{ds} \Big|_{s=t} \tau(S_1) \right] \otimes \tau(S_2) \Big|_{s=t} + \tau(S_1) \Big|_{s=t} \otimes \frac{d}{ds} \Big|_{s=t} \tau(S_2)$$

$$= (\nabla_{\xi} S_1) \otimes S_2 + S_1 \otimes \nabla_{\xi} S_2 \quad (A)$$

* Eq. (C) implies that ∇ and contractions do commute.

Action of ∇ on tensors in a chart?

Recall: $\nabla_{\xi} \frac{\partial}{\partial x^i} = \xi^l \Gamma^k_{li} \frac{\partial}{\partial x^k}$

Problem: Find $\nabla_{\xi} dx^i = ?$

Consider $\eta \otimes \omega$.

$\downarrow$ tangent vector field $\downarrow$ cotangent vector field

Differentiate:

$$\nabla_{\xi} (\eta \otimes \omega) \stackrel{(A)}{=} (\nabla_{\xi} \eta) \otimes \omega + \eta \otimes \nabla_{\xi} \omega$$

Contract: (use that ∇_{ξ} and contraction commute)

(by exercise above) $\xi(\omega(\eta)) = \nabla_{\xi}(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)$

$\underbrace{\hspace{1cm}}$ scalar function

Same strategy will be used below for general tensors.

(i.e. $\xi(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)$)

$\Rightarrow$ An expression for $\nabla_{\xi}(\omega)(\eta)$:

$$(\nabla_{\xi} \omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_{\xi} \eta) \quad (*)$$

Now: Choose $\omega := dx^i$ and $\eta := \frac{\partial}{\partial x^i}$

$$\begin{aligned} \Rightarrow (\nabla_{\xi} dx^i) \left(\frac{\partial}{\partial x^i} \right) &= \xi \left(\underbrace{\left\langle dx^i, \frac{\partial}{\partial x^i} \right\rangle}_{=0} \right) - \left\langle dx^i, \nabla_{\xi} \frac{\partial}{\partial x^i} \right\rangle \\ &= - \left\langle dx^i, \xi^l \Gamma^k_{li} \frac{\partial}{\partial x^k} \right\rangle \\ &= - \xi^l \Gamma^i_{li} \end{aligned}$$

Notation:
 $\langle \omega, \xi \rangle = \omega(\xi)$
 (inner product, contraction)

$$\Rightarrow \nabla_{\xi} dx^i = - \xi^l \Gamma^i_{li} dx^i$$

For general tensors: (by exactly same strategy as above but applied to multiple tensor products, we obtain:

$$\nabla_{\xi} S(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_s) \quad (\text{as in Eq. (*) above})$$

$$= S'(S(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_s))$$

$$- S'(\nabla_{\xi} \eta_1, \eta_2, \dots, \eta_r, \omega_1, \dots, \omega_s) - \dots$$

$$- S'(\eta_1, \dots, \nabla_{\xi} \eta_r, \omega_1, \dots, \omega_s)$$

$$- S'(\eta_1, \dots, \eta_r, \nabla_{\xi} \omega_1, \omega_2, \dots, \omega_s) + \dots$$

$$- S'(\eta_1, \dots, \eta_r, \omega_1, \dots, \nabla_{\xi} \omega_s)$$

Choosing the basis vectors dx^i and $\frac{\partial}{\partial x^i}$, we obtain for

$$S = S_{j_1, \dots, j_q}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

that $\nabla_{\xi} S$ reads

$$\nabla_{\xi} S = \xi^k S_{j_1, \dots, j_q, k}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

with:

$$\begin{aligned} S_{j_1, \dots, j_q, k}^{i_1, \dots, i_p} := & S_{j_1, \dots, j_q, k}^{i_1, \dots, i_p} + \Gamma_{k\ell}^{i_1} S_{j_1, \dots, j_q}^{\ell i_2 \dots i_p} \\ & + \dots + \Gamma_{k\ell}^{i_p} S_{j_1, \dots, j_q}^{i_1 \dots \ell} \\ & - \Gamma_{kj}^{\ell} S_{\ell \dots j_q}^{i_1 \dots i_p} \\ & - \dots - \Gamma_{kj}^{\ell} S_{j_1 \dots \ell}^{i_1 \dots i_p} \end{aligned}$$

Special cases:

▣ Tangent vector fields:

$$\xi^i_{;j;k} = \xi^i_{,k} + \xi^j \Gamma^i_{kj}$$

▣ Cotangent vector fields:

$$\omega_{i;j;k} = \omega_{i,k} - \omega_j \Gamma^l_{ki}$$

Recall: Specifying ∇ specifies parallel transport of vectors and this should specify the manifold's shape, but how?

→ Indeed, ∇ specifies Torsion & Curvature