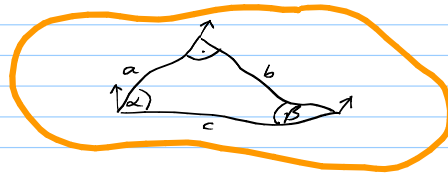
Note Title

Recall: The nontrivial shape of a manifold reveals itself in several ways:



1. Violation of angle sum law, $\alpha + \beta + 90° \neq 180°$.
   ⇝ Can encode shape through deficit angles (used in some quantum gravity approaches)

2. Violation of Pythagoras' law, $a^2 + b^2 \neq c^2$.
   ⇝ Can encode shape through metric distances: $(\mathcal{M}, g)$

3. Nontrivial parallel transport of vectors on loops.
   ⇝ Can encode shape through affine connection: $(\mathcal{M}, \Gamma)$

This makes it hard to identify the true degrees of freedom, so that they can be quantized.

Observe: Such local descriptions carry redundant information!

Why? Two (pseudo-)Riemannian mflds $(\mathcal{M}, g), (\mathcal{M}, g')$ must be considered equivalent, i.e., they are describing the same space(-time), if there exists an isometric, i.e., metric-preserving, isomorphism:

$$\boxed{\varphi: (\mathcal{M}, g) \rightarrow (\mathcal{M}, g')}$$

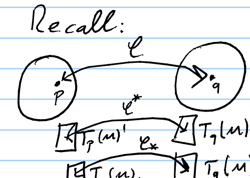
Here: φ is called metric-preserving if, under the pull-back map

$$T\varphi^*: T_p(\mathcal{M})_2 \rightarrow T_p(\mathcal{M})_2$$

the metric obeys:

$$T\varphi_*(g) = g'$$

Recall:



⇝ φ can then be considered to be a mere change of chart.

Intuition: (M,g), (M,g') that are related by an isometric diffeomorphism are mere cd changes of another, i.e., have the same "shape".

Definition: A (pseudo-) Riemannian structure, say Ξ, is an equivalence class of (pseudo-) Riemannian manifolds which can be mapped into each other via metric-preserving diffeomorphisms, i.e., via changes of coordinates.

$\Rightarrow$ Space(-time) will need to be modelled as a (pseudo-) Riemannian structure, Ξ, i.e., as an equivalence class of pairs (M,g).

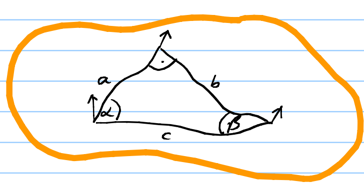
Problem: These equiv. classes are hard to handle because absence or existence of φ is hard to check!

$\Rightarrow$ One would like to be able to reliably identify exactly one representative (M,g) per class Ξ.

- This would be called a "fixing of gauge".
- Why would this be useful?

A key example of when gauge fixing needed: Quantum gravity

We discussed detecting and describing shape through



- deficiency angles
- nontrivial metric distances (M,g)
- nontrivial parallel transport (M,Γ)

Recall: Quantum theory can be formulated in path integral form.

Applied to gravity:

Expect to have to handle path integrals of the type:

$$\int_{\text{"all Riemannian structures } \Xi\text{"}} e^{iS'(\Xi)}\, D\Xi$$

But what we initially have is, roughly of the form:

$$\int_{\text{"all } g\text{"}} e^{iS'(g)}\, \delta(?)\, Dg \quad \text{or} \quad \int_{\text{"all } \Gamma\text{"}} e^{iS'(\Xi)}\, \delta(?)\, D\Gamma$$

Here, $\delta(?)$ should be such that from each equivalence class of the g's or the Γ's only exactly one contributes to the path integral.

$\rightsquigarrow$ Much of Quantum Gravity research is concerned with working out suitable $\delta(?)$ for g's or Γ's or other variables formed from them, such as the frame fields (see "Loop quantum gravity").

Q: Can one detect and describe a (pseudo-) Riemannian structure Ξ directly?

A: Possibly yes, using "Spectral Geometry":

Idea: A manifold's vibration spectrum $\underbrace{\{\lambda_n\}}_{\text{Independent of coordinate systems!}}$ depends only on Ξ!

Key question of the field of spectral geometry: (Weyl 1911)

$$\boxed{\text{Does the spectrum } \{\lambda_n\} \text{ encode all about the shape, i.e., } \Xi \text{ ?}}$$

Remarks:

- It cannot, if $\mathcal{M}$ has infinite volume, because then the spectrum of Δ will become (almost) completely continuous.

- The spectral geometry of pseudo-Riemannian manifolds is still very little developed.

Theorem:

- Assume $(\mathcal{M}, g)$ is a compact (implies finite volume) Riemannian manifold without boundary, $\partial\mathcal{M} = \phi$.

- Then, each $\mathrm{spec}(\Delta_p)$ is discrete, with finite degeneracies and without accumulation points.

In practice:

We can describe any arbitrarily large part of the universe by a compact Riemannian manifold, $(\mathcal{M}, g)$.

This allows us to describe, e.g., 3-dim. space at any fixed time (or also 4-dim. spacetime after so-called Wick rotation).

Types of waves (incl. sounds) on $\mathcal{M}$: assumed compact, no boundary

Consider p-form fields $\omega(x)$ on $\mathcal{M}$, with time evolution, e.g.,:

1. Schrödinger equation: $i\hbar\partial_t\,\omega(x,t) = -\frac{\hbar^2}{2m}\Delta_p\,\omega(x,t)$

2. Heat equation: $\partial_t\,\omega(x,t) = -\alpha\,\Delta_p\,\omega(x,t)$

3. Klein Gordon (and acoustic) eqn: $-\partial_t^2\,\omega(x,t) = \beta\,\Delta_p\,\omega(x,t)$

- Each of them can be solved via separation of variables:
- Assume we find an eigenform $\tilde{\omega}(x)$ of Δ on $\mathcal{M}$:

$$\boxed{\Delta_p \tilde{\omega}(x) = \lambda \tilde{\omega}(x)}$$

- They exist: Each Δ is self-adjoint, w.r.t. the inner product $(\omega, \nu) = \int_{\mathcal{M}} \omega \wedge * \nu$.

Then: Schrödinger eqn solved by: $\omega(x,t) := e^{\frac{i\hbar}{2m}\lambda t}\,\tilde{\omega}(x)$

Heat eqn solved by: $\omega(x,t) := e^{-\alpha\lambda t}\,\tilde{\omega}(x)$

Klein Gordon eqn solved by: $\omega_\pm(x,t) := e^{\pm i\sqrt{\beta\lambda}\,t}\,\tilde{\omega}(x)$

$\Rightarrow$ The spectrum $\mathrm{spec}(\Delta_p)$ is the overtone spectrum of p-form type waves on the manifold $\mathcal{M}$.

Properties of $\mathrm{spec}(\Delta_p)$:

- Expectation:

  The spectra $\mathrm{spec}(\Delta_p)$ for different p carry different information about $\mathcal{M}$:

  E.g., scalar and vector seismic waves travel (and reflect) differently.

- But recall also: a) $[\Delta, *] = 0$

  b) $[\Delta, d] = 0$

  c) $[\Delta, \delta] = 0$

  This will relate $\mathrm{spec}(\Delta_p)$ to $\mathrm{spec}(\Delta_{n-p})$, $\mathrm{spec}(\Delta_{p+1})$ and $\mathrm{spec}(\Delta_{p-1})$:

Use $[\Delta, *] = 0$:

Assume: $\omega \in \Lambda_p$ and $\Delta\omega = \lambda\omega$.

Define: $\nu := *\omega \in \Lambda_{n-p}$

Then:

$$\Delta\nu = \Delta * \omega = * \Delta\omega = * \lambda\omega = \lambda\nu$$

$$\Rightarrow \boxed{spec(\Delta_p) = spec(\Delta_{n-p})}$$

Next:

Careful utilization of $[\Delta, d] = 0$ and $[\Delta, \delta] = 0$ yields much more information about these spectra!

- **Notice that:** Δ maps exact forms $\omega = d\nu$ into exact forms:

$$\Delta\omega = \Delta d\nu = \overbrace{d\Delta\nu}^{\text{an exact form}}$$

i.e.: $\boxed{\Delta : d\Lambda_r \to d\Lambda_r}$

$d\Lambda_r$ = image of Λ_r under d.

- **Analogously:** Δ maps co-exact forms $\omega = \delta\beta$ into co-exact forms:

$$\Delta\omega = \Delta\delta\beta = \overbrace{\delta\Delta\beta}^{\text{a co-exact form}}$$

i.e.: $\boxed{\Delta : \delta\Lambda_r \to \delta\Lambda_r}$

- **Also:** Δ can map forms into 0, namely its eigenspace with eigenvalue 0, denoted Λ_r^0. Λ_r^0 is called the space of "**harmonic**" p-forms.

$$\boxed{\Delta : \Lambda_r^0 \to 0}$$

Thus: Δ maps $d\Lambda_r$ and $\delta\Lambda_r$ and Λ_r^0 into themselves.

Are there any other forms that Δ could act on? No!

Proposition ("Hodge decomposition"):

$$\boxed{\Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^0}$$

(Recall that $\oplus$ implies that the three spaces are orthogonal!)

Q: Why useful?

A: It means that every eigenvector of Δ_p is either in $d\Lambda_{p-1}$ or in $\delta\Lambda_{p+1}$ or in Λ_p^0 but is never a linear combination of vectors in these spaces.

Proof: It is clear that $d\Lambda_{p-1} \subset \Lambda_p$ and $\delta\Lambda_{p+1} \subset \Lambda_p$.

We need to show the orthogonalities and completeness:

□ Show that $d\Lambda_{p-1} \perp \delta\Lambda_{p+1}$:

Indeed, assume $\omega = d\nu \in \Lambda_p$ and $\alpha = \delta\beta \in \Lambda_p$.

Then: $(\omega, \alpha) = (d\nu, \delta\beta) \underset{-d^+=\delta}{\overset{use}{=}} (\overset{0}{\overbrace{dd}}\nu, \beta) = 0$ ✓

Exercise: study the remainder of the proof.

□ Show that if $\omega \in \Lambda_p$ and $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$ then: $\omega \in \Lambda_p^0$.

Indeed, assume $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$. Then:

$\forall\alpha$: $(d\alpha, \omega) = 0$ i.e. $-(\alpha, \delta\omega) = 0$ $\Rightarrow$ $\delta\omega = 0$

$\forall\beta$: $(\delta\beta, \omega) = 0$ i.e. $-(\beta, d\omega) = 0$ $\Rightarrow$ $d\omega = 0$

$\Rightarrow$ $\Delta\omega = (d\delta + \delta d)\omega = 0$ $\Rightarrow$ $\omega \in \Lambda_p^0$ ✓

Show that if $\omega \in \Lambda^0_p$ then $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$.

Assume $\omega \in \Lambda^0_p$, i.e., $\Delta\omega = 0$, i.e., $(\delta d + d\delta)\omega = 0$.

$\Rightarrow (\omega, (d\delta + \delta d)\omega) = 0$

$\Rightarrow \overbrace{(\delta\omega, \delta\omega)}^{\geq 0} + \overbrace{(d\omega, d\omega)}^{\geq 0} = 0 \Rightarrow \delta\omega = 0$ and $d\omega = 0$.

(I.e., harmonic forms are closed and co-closed but not exact or co-exact.
Thus, $B_p := \dim(\Lambda^0_p)$ measures topological nontriviality.
The B_p are called the "Betti numbers".)

$\Rightarrow \forall \alpha \in \Lambda_{p-1}: \quad (\alpha, \delta\omega) = 0$, i.e., $(d\alpha, \omega) = 0$.

$\Rightarrow \omega \perp d\Lambda_{p-1}$ ✓

Also: $\forall \beta \in \Lambda_{p+1}: (\beta, d\omega) = 0$, i.e., $(\delta\beta, \omega) = 0$.

$\Rightarrow \omega \perp \delta\Lambda_{p+1}$ ✓

<u>Conclusion so far:</u>

In the Hodge decomposition, Δ maps every term into itself, i.e., Δ can be diagonalized in each $d\Lambda_r$, $\delta\Lambda_r$, Λ^0_r separately.

$$\begin{cases} \quad\vdots \\ \Lambda_{p-1} = d\Lambda_{p-2} \oplus \delta\Lambda_p \oplus \Lambda^0_{p-1} \\ \Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda^0_p \\ \Lambda_{p+1} = d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \Lambda^0_{p+1} \\ \quad\vdots \end{cases}$$

$\Rightarrow$ Δ has eigenvectors and -values on each of these subspaces, for all r:

$$\mathrm{spec}\left(\Delta\big|_{d\Lambda_r}\right), \quad \mathrm{spec}\left(\Delta\big|_{\delta\Lambda_r}\right), \quad \mathrm{spec}\left(\Delta\big|_{\Lambda^0_r}\right) = \{0\} \ \ldots$$

These spectra are related!

Proposition: $\operatorname{spec}\left(\Delta\big|_{d\Lambda_r}\right) = \operatorname{spec}\left(\Delta\big|_{\delta\Lambda_{r+1}}\right)$

and for each eigenvector in one there is one in the other.

This means:

$$\vdots$$

$$\Lambda_{p-1} = d\Lambda_{p-2} \oplus \delta\Lambda_p \oplus \mathring{\Lambda}_{p-1}$$

same spectrum

$$\Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \mathring{\Lambda}_p$$

same spectrum

$$\Lambda_{p+1} = d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \mathring{\Lambda}_{p+1}$$

$$\vdots$$

Proof:

Assume: $\lambda \in \operatorname{spec}\left(\Delta\big|_{d\Lambda_r}\right)$ with eigenvector $w \in d\Lambda_r$.

Define: $v := \delta w \in \delta\Lambda_{r+1}$

Then: $\Delta v = \Delta\delta w = \delta\Delta w = \lambda\delta w = \lambda v$

$\Rightarrow \lambda \in \operatorname{spec}\left(\Delta\big|_{\delta\Lambda_{r+1}}\right)$ and v is the eigenvector.

Conversely:

Assume: $\lambda \in \operatorname{spec}\left(\Delta\big|_{\delta\Lambda_{r+1}}\right)$ with eigenvector $w \in \delta\Lambda_{r+1}$.

Define: $v := dw \in d\Lambda_r$

Then: $\Delta v = \Delta dw = d\Delta w = \lambda dw = \lambda v$

$\Rightarrow \lambda \in \operatorname{spec}\left(\Delta\big|_{d\Lambda_r}\right)$ and v is the eigenvector. ✓

Re-use $[\Delta, *] = 0$:

Proposition: $* : \boxed{d\Lambda_r \to \delta \Lambda_{n-r}}$ (with $d\Lambda_r \subseteq \Lambda_{r+1}$, $\delta\Lambda_{n-r} \subseteq \Lambda_{n-r-1}$)

i.e.: $*$: exact $r+1$ forms $\to$ co-exact $n-r-1$ forms

Proof: Assume $\omega = d\rho \in d\Lambda_r$

Define $\nu := *\omega$

$$\Rightarrow \nu = *d\rho = (-1)^{r(n-r)} \underbrace{*d*}_{=\delta} *\rho$$

$$= \delta\alpha \in \delta\Lambda_{n-r} \quad \text{for } \alpha = (-1)^{r(n-r)} *\rho$$

Proposition: $* : \boxed{\delta\Lambda_r \to d\Lambda_{n-r}}$

Proof: Exercise.

Recall: $*$ preserves the spectrum of Δ as we showed already.

$\Rightarrow$

Summary:

$$\vdots$$

$$\Lambda_{p-1} = d\Lambda_{p-2} \oplus \delta\Lambda_p \oplus \Lambda^0_{p-1}$$

same spectrum

$$\Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda^0_p$$

same spectrum

$$\Lambda_{p+1} = d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \Lambda^0_{p+1}$$

$$\vdots$$

Now we also found:

$$\Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^0$$

same   same spectrum   spectrum

$$\vdots$$

$$\Lambda_{n-p} = d\Lambda_{n-p-1} \oplus \delta\Lambda_{n-p+1} \oplus \Lambda^0_{n-p}$$

Example: $\dim(\mathcal{M})=3$ Exercise: do same for $\dim(\mathcal{M})=4$

$$\Lambda_0 = \delta\Lambda_1 \oplus \Lambda_0^\circ$$

$$\Lambda_1 = d\Lambda_0 \oplus \delta\Lambda_2 \oplus \Lambda_1^\circ$$

$$\Lambda_2 = d\Lambda_1 \oplus \delta\Lambda_3 \oplus \Lambda_2^\circ$$

$$\Lambda_3 = d\Lambda_2 \oplus \Lambda_3^\circ$$

Same color means same spectrum of Δ.

Conclusion: There is relatively little independent information in the spectra of p-form waves on $\mathcal{M}$!
E.g., when $\dim(\mathcal{M})=3$, then the spectrum of co-vector waves $\operatorname{spec}(\Delta|_{\Lambda_1})$ has already all information of all these spectra.

Literature: (neglecting literature on detecting boundary shapes from spectra)

Indeed: The spectra of Δ do not contain sufficient information in general to uniquely identify the Riemannian structure from the spectra alone:

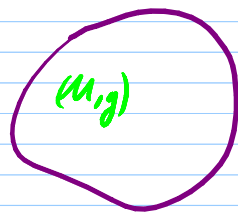
Examples: Cases have been found of pairs $(\mathcal{M}, g)$, $(\tilde{\mathcal{M}}, \tilde{g})$ that are isospectral for Δ on all Λ_p but that are not diffeomorphically isometric!

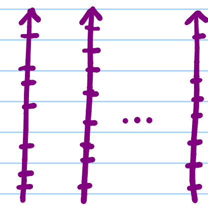
Nevertheless: All examples are of limited significance:
- manifolds that are locally if not globally isometric, or
- manifolds that are isospectral only w. respect to some Δ or
- manifolds that are discrete pairs (e.g. mirror images).

Fresh approach to spectral geometry (AK)

Strategy: Iterate infinitesimal inverse spectral geometry

Assume both, the mfld and its spectra are given:





A compact Riemannian manifold (M,g) without boundary

The spectra $\{\lambda_n^{(i)}\}$ of Laplacians $\Delta^{(i)}$ on the manifold.

Could be Laplacians not only on forms but also on general tensors.

Perturbation:

Now change the shape of (M,g) slightly, through:

$$g \to g + h$$

This will slightly change the spectra to

$$\{\lambda_n^{(i)}\} \longrightarrow \{\lambda_n^{(i)} + \mu_n^{(i)}\}$$

Why is this linearization useful?

- One can define a self-adjoint Laplacian $\Delta^{(m)}$ on $T_2(M)$, with Hilbert basis $\{b_n(x)\}$ and eigenvalues $\{\lambda_n^{(m)}\}$:

$$\Delta^{(m)} b_n(x) = \lambda_n b_n(x)$$

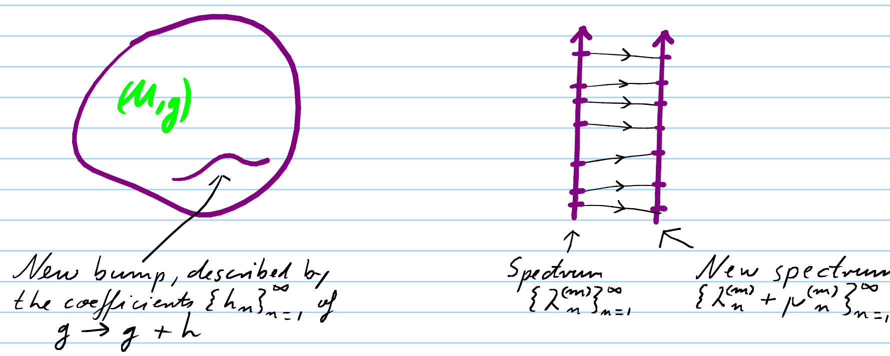
⇒ The metric's perturbation $h \in T_g(M)$ can be expanded:

$$h = \sum_{n=1}^{\infty} h_n b_n(x)$$

The perturbation of $\mathrm{spec}(\Delta^{(m)})$ is:

$$\{\lambda_n^{(m)}\} \to \{\lambda_n^{(m)} + \mu_n^{(m)}\}$$

⇒



⇒ We obtain a linear map S:

$$S: \{h_n\} \longrightarrow \{\mu_n\}$$

$$S': h_n \to \mu_n = S_{nm} h_m$$

Notice: Consider only eigenvectors and eigenvalues up to a cutoff scale. Then, there are as many parameters $\{h_n\}_{n=1}^{N}$ as $\{\mu_n\}_{n=1}^{N}$.

⇒ S is a square matrix.

If $\det(S') \neq 0$, then S'^{-1} exists.

⇝ should be able to iterate the perturbations?

This is ongoing research.

Remarks:

- Not all h actually change the shape:

  Iff $h = L_\xi g$ for some vector field ξ, then $g \to g + h$ is merely the infinitesimal change of chart belonging to the flow induced by ξ.

- Symmetric covariant 2-tensors such as h have a canonical decomposition similar to the Hodge decomposition. Thus, Δ has three spectra on $T_2(M)$.

Reference: See also e.g. the video of my talk at PI: http://pirsa.org/15090062

Infinitesimal spectral geometry arose from my paper on how Spacetime could be simultaneously continuous and discrete, in the same way that information can.