Note Title

Plan: I The dynamics of matter & radiation in curved spacetime

II Energy-momentum tensor

III The dynamics of spacetime itself.

1. Recall: On a (pseudo)-Riemannian mfld, equations are well-defined only if defined independently of any chart.

$\Rightarrow$ Any eqn, including the eqns of motions for matter fields must be eqns among tensors and their covariant derivatives.

$\Rightarrow$ Need a tensor field, ψ, for each species of particle:

e^-, q, gluon, $\pi^\pm$, photon, $W^\pm$, etc...

Notation: $\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$ — $a\ldots b$: contravariant; $c\ldots d$: covariant; (i): species label

Note: any spinor equation can also be expressed as a (complicated) tensor equation (see e.g. Hawking & Ellis, p 59)

Question:

Could we have also an additional connection field $\tilde{\Gamma}^k_{ij}$?

Yes, we could: But, the difference field $Q^k{}_{ij} := \Gamma^k{}_{ij} - \tilde{\Gamma}^k{}_{ij}$ is actually a tensor field!

$$\Gamma^r{}_{ab} \longrightarrow \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij}$$

$$\tilde{\Gamma}^r{}_{ab} \longrightarrow \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\tilde{\Gamma}^k{}_{ij}$$

$$\Rightarrow \quad \left(\Gamma^r{}_{ab} - \tilde{\Gamma}^r{}_{ab}\right) \longrightarrow \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\left(\Gamma^k{}_{ij} - \tilde{\Gamma}^k{}_{ij}\right)$$

$$\Rightarrow \quad \boxed{Q^r{}_{ab} \longrightarrow \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b} Q^k{}_{ij}}$$

i.e. $Q^r{}_{ab}$ is a tensor due to having the correct transformation property according to the physicist's definition of a tensor.

$\Rightarrow$ Introducing an additional connection $\tilde{\Gamma}$ is same as introducing simply a new tensor field Q.

Remark: $\Longrightarrow$ "variations" $\delta\Gamma^r{}_{ab}$ will behave tensorially!

Eqns of motion of matter fields?

Action principle: (As in special relativity)

Any theory of matter fields can be defined by specifying the so-called Lagrangian function, L, namely a scalar function of the matter fields $\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$ and their first covariant derivatives, and now also of the metric g:

I we'll sometimes omit the indices

$$\boxed{L(\psi) = L^{(matter)}\left(\{\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\}, \{\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\}, g\right)}$$

Define the action functional:

$$S[\psi] := \int_B \underbrace{\overbrace{L(\psi)}^{\text{scalar}} \, \overbrace{\sqrt{g} \, d^4x}^{\Omega = \text{volume form}}}_{n\text{-form}} \in \mathbb{R}$$

← some bounded and closed 4-dim region in M.

Thus, each physical field $\psi(x,t)$ (as a function of both space <u>and time</u>) is mapped into a number $S[\psi]$.

Action principle (or postulate) of classical physics:

In nature, physical fields ψ are such that $S[\psi]$ is extremal in the space of all fields ψ.

Thus: The matter fields ψ obey:

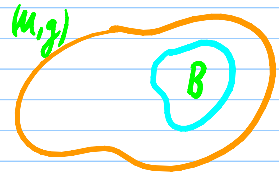
$$\boxed{\frac{\delta S[\psi]}{\delta \psi} = 0} \qquad (*)$$

These will be the eqns of motion for the fields ψ.

Definition of $(*)$?

Def: A "variation $\delta\psi$" of the fields $\psi_{(i)}(p)$ (with $p \in B \subset M$) in a region B is a one-parameter deformation, $\psi_{(i)}(\lambda, p)$, with $\lambda \in \overbrace{(-\varepsilon, \varepsilon)}^{\text{some finite interval}}$,

↑ deformation parameter



so that

i.e. $\lambda=0$ is non-deformation

1.) $\Psi_{(i)}(0,p) = \Psi_{(i)}(p) \quad \forall p \in M$

2.) $\Psi_{(i)}(\lambda,p) = \Psi_{(i)}(p) \quad \forall \lambda$, if $p \in M-B$

i.e. no deformation at all outside region B.

Def: Then, we define:

$$\delta \Psi_{(i)}(p) := \left.\frac{\partial \Psi_{(i)}(\lambda,p)}{\partial \lambda}\right|_{\lambda=0}$$

Def: The action principle now reads:

$$\boxed{0 = \left.\frac{\partial S[\Psi]}{\partial \lambda}\right|_{\lambda=0} \quad \text{for all variations } \delta\Psi_{(i)}.}$$

Evaluate:

$$0 = \left.\frac{\partial S}{\partial \lambda}\right|_{\lambda=0} = \sum_i \int_B \Bigg[\overbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\dots b}{}_{c\dots d}} \underbrace{\delta \Psi_{(i)}{}^{a\dots b}{}_{c\dots d}}_{\text{recall: } = \left.\frac{d\Psi_{(i)}{}^{a\dots b}{}_{c\dots d}}{d\lambda}\right|_{\lambda=0}}}^{\text{Term I}}$$

$$+ \overbrace{\underbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}}}_{\substack{\text{by assumption,} \\ L \text{ depends also on} \\ \text{the 1st cov. derivatives.}}} \delta\left(\Psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}\right)}^{\text{Term II}} \Bigg] \sqrt{g}\, d^4x$$

Evaluate terms I, II separately:

Term II:

We notice:

Recall: At origin of geodesic coordinate system, $\Gamma^k{}_{ij}=0$, i.e. $\psi_{;e}=\psi_{,e}$. But then $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \lambda}$ commute. True in any coordinate system.

$$\delta\left(\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)=\left(\delta\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e}$$

$$\Rightarrow \text{Term II} = \sum_i \int_B \frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\left(\delta\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e}\sqrt{g}\, d^4x$$

$$= \sum_i \int_B \Bigg[\Bigg(\overbrace{\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\,\delta\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{=:K^e}\Bigg)_{;e}$$

(use Leibniz rule to verify)

$$- \left(\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e}\delta\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\Bigg]\sqrt{g}\, d^4x$$

One term is a "boundary term":

$$\sum_i \int_B K^e{}_{;e}\sqrt{g}\, d^4x$$

Exercise:

show that for all ξ^μ:

$$\xi^\mu{}_{;\mu}\,\Omega = \mathrm{div}_\Omega \xi$$

if $\Omega = \sqrt{g}\, dx^1\wedge\ldots\wedge dx^n$

$$= \sum_i \int_B \mathrm{div}_\Omega K$$

Gauß' theorem $\Rightarrow$

$$= \sum_i \int_{\partial B} \underbrace{i_K}_{\text{inner derivation}}\Omega$$

(Recall: $\mathrm{div}_\Omega K = L_K\Omega$
$= (i_K\circ d + d\circ i_K)\Omega$
$= d\circ i_K\Omega$)

but: $K\propto \delta\psi$ and $\delta\psi(p)=0$ if $p\in\partial B$
by property 2) of variations.

$$\Rightarrow \quad = 0 \quad !$$

Thus, term II simplifies and we obtain:

$$0 = \frac{\partial S}{\partial \lambda}\Big|_{\lambda=0} = \sum_i \int_B \Big[\overbrace{\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} \delta \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{\text{Term I}} - \overbrace{\left(\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e} \delta \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{\text{Term II}} \Big] \sqrt{g}\, d^4x$$

Since must hold for all variations $\delta\psi$

$\Rightarrow$

$$\boxed{\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} - \left(\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e} = 0}$$

"Euler-Lagrange equations"

Given $L(\psi)$, these eqns yield the eqns. of motion for ψ.

<u>Example</u>: A real-valued scalar field ψ ← real-valued

- Such ψ describe e.g.:
  - π^0 meson (quark+antiquark)
  - inflaton
- Lagrangian?
  - Choose geodesic cds at arb. point and appeal to equiv. principle.
  - Obtain from spec. relat. Lagrangian:

$$L = -\frac{1}{2}\left(\psi_{;a}\psi_{;b}g^{ab} + \frac{m^2}{\hbar^2}\psi^2\right)$$

- Euler-Lagrange equation: Klein-Gordon equation

(Exercise: verify)

$$\boxed{\psi_{;ab}\, g^{ab} - \frac{m^2}{\hbar^2}\psi = 0}$$

Example: The electromagnetic fields

- Assume there are no charges (i.e. there are only EM waves)
- Define the "EM 4-potential" as a real-number-valued one-form A.
- Consider the field strength tensor F:

$$F := dA$$

- Recall that the E and B fields are components of the 2-form F. (up to a factor of 2)

- The Lagrangian (from equiv. principle):

$$L = -\frac{1}{16\pi} F_{ab} F_{cd}\, g^{ac} g^{bd}$$

(Exercise: write in terms of forms)

- Varying w. resp. to A, the E.L. equations read:

$$\boxed{F_{ab;c}\, g^{bc} = 0}$$

recall: this is $\delta F = 0$

"Maxwell eqns".

- It is also true that

$$F_{ab;c} + F_{ca;b} + F_{bc;a} = 0$$

but this is not an Euler Lagrange eqn. It is simply: $\boxed{dF = 0}$ (which holds because $F = dA$ and $d^2 = 0$)

Example: A charged scalar field ψ, ← complex-valued

(such ψ describe, e.g., $\pi^\pm$ mesons)

together with electromagnetism.

Equiv. principle yields from spec. relativity:

Why ψ complex?
Mixed term is Lorentz force.
If ψ was real, it would be absent:

$$-ieA_a\psi^*\psi_{;b}g^{ab}$$
$$+ie\,A_b\psi^*_{;a}\psi\, g^{ab}$$
$$=ieA_a g^{ab}(\psi^*_{;a}\psi-\psi_{;a}\psi^*)$$
$$=0 \text{ if } \psi^*=\psi$$

electric charge constant

$$L=-\frac{1}{2}\left(\psi^*_{;a}-ieA_a\psi^*\right)\left(\psi_{;b}+ieA_b\psi\right)g^{ab}$$
$$-\frac{1}{2}\frac{m^2}{\hbar^2}\psi^*\psi-\frac{1}{16\pi}F_{ab}F_{cd}\,g^{ac}g^{bd}$$

Vary w. resp. to ψ^* $\Rightarrow$ E.L. eqn:

$$\underbrace{\psi_{;ab}\,g^{ab}-\frac{m^2}{\hbar^2}\psi}_{\text{Klein Gordon part}}+\underbrace{ieA_a g^{ab}\left(\psi_{;b}+ieA_b\psi\right)+ieA_{a;b}\,g^{ab}\psi}_{\psi \text{ is affected by } A}=0$$

and varying w. resp. to ψ yields the compl. conj. equation.

Vary w. resp. to A_a $\Rightarrow$ E.L. eqn:

$$\underbrace{\frac{1}{4\pi}F_{ab;c}\,g^{bc}}_{\text{plain Maxwell part}}\underbrace{-ie\psi\left(\psi^*_{;a}-ieA_a\psi^*\right)+ie\psi^*\left(\psi_{;a}+ieA_a\psi\right)}_{A \text{ is affected by } \psi,\psi^*.}=0$$

Dirac equation: (Brief treatment of basics only of Dirac spinors)

In special relativity: (with units such that $\hbar = 1$)

$$\boxed{\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0}$$

"Dirac equation" (D)

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ is a "Spinor"

describes spin $1/2$ particles such as electrons and quarks

and the four 4×4 matrices γ^μ obey:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \quad (*)$$

$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{\mu\nu}$

Why (*)? Equation (*) is specifically chosen so that each component of ψ obeys the Klein Gordon equation. Indeed:

$$(D) \Rightarrow \left(-i\gamma^\mu\partial_\mu - m\right)\left(i\gamma^\nu\partial_\nu - m\right)\psi = 0$$

$$\Rightarrow \left(+\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + i\gamma^\mu\partial_\mu m - im\gamma^\nu\partial_\nu + m^2\right)\psi = 0$$

symmetric under $\mu \leftrightarrow \nu$

$$\Rightarrow \left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2\right)\psi = 0$$

antisymmetric part not needed, it would drop out.

$$\Rightarrow \left(\frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\mu\partial_\nu + m^2\right)\psi = 0$$

$$\overset{(*)}{\Rightarrow} \mathbb{1}\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\psi = 0$$

which is the Klein Gordon equation in flat space.

In general relativity:

- By choosing an orthonormal tetrad, $\{\theta^i\}$, we achieve

$$g^{\mu\nu} = \eta^{\mu\nu} \quad \forall p \in M$$

i.e. one set of matrices γ^μ obeying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ suffices.

- This motivates:

$$(i\gamma^\mu \nabla_{e_\mu} - m)\,\psi = 0$$

- But what is the covariant derivative of a spinor?

$$\nabla_{e_\mu} \psi = ?$$

Recall: The covariant derivative of a vector yields the infinitesimal Lorentz transformation by which the vector rotates under infinitesimal parallel transport.

Idea: The covariant derivative of a spinor should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

Recall: Infinitesimal parallel transport of a vector e_ρ in direction e_μ:

$$e_\rho \to e_\rho + \nabla_{e_\mu} e_\rho = e_\rho + \omega^\sigma_{\ \rho}(e_\mu)\, e_\sigma$$

Recall: the curvature 1-form takes values that are infinitesimal Lorentz transformations.

This is an infinitesimal Lorentz transformation $\Lambda_\rho^{\ \sigma}$:

$$e_\rho \to \Lambda_\rho^{\ \sigma} e_\sigma \quad \text{with} \quad \Lambda^\sigma_{\ \rho} = \delta^\sigma_{\ \rho} + \omega^\sigma_{\ \rho}(e_\mu)$$

because $\omega^\sigma_{\ \rho}$ obeys: $\omega_{\sigma\rho} = -\omega_{\rho\sigma}$. (Which is the defining equation for infinitesimal Lorentz transformations)

Recall intuition why parallel transport yields Lorentz transformation: Parallel transport preserves the lengths of vectors, i.e. they can at most "rotate" and in 3+1 dim. this is Lorentz transformations.

Now that we know the inf. Lorentz transf. for any inf. parallel transport:

Strategy: Apply the same inf. Lorentz transformation on spinors for their parallel transport.

To this end: Recall from Special Relativity how an infinitesimal Lorentz transformation acts on a spinor:

- Assume $\{s_i\}_{i=1}^{4}$ are ON basis in Spinor space, i.e.

$$\psi = \psi^i(x)\, s_i$$

these are Spinor indices: $i = 1,2,3,4$

- How do the s_i transform under Lorentz transformations? I.e., what is $\nabla_{e_i} s_j = ?$ (In analogy to $\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a) e_\nu$)

- From special relativity it is known that under infinitesimal Lorentz transformations,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

vectors transform as

$$e_\mu \to e_\mu + \omega_\mu{}^\nu e_\nu$$

and the Dirac spinors transform as:

$$s_i \to s_i \underbrace{- \frac{1}{4}\omega_\mu{}^\nu [\gamma^\mu, \gamma_\nu]\, s_i}$$

$\Rightarrow$ under infinitesimal Lorentz transf. the spinor "rotates" by this amount.

Where does $[\gamma^\mu, \gamma^\nu]$ come from?

Recall that e.g. translations in space are generated by momentum operators, $e^{-i\vec{\alpha}\vec{p}} f(x) e^{i\vec{\alpha}\vec{p}} = f(x+\vec{\alpha})$, if they obey the commutation relations $[x_i, p_j] = i\delta_{ij}$.

Similarly, Lorentz transformations are generated by operators $M^{\mu\nu}$:

$$e^{-i\omega_{\mu\nu}M^{\mu\nu}}\, f\, e^{i\overset{\text{antisym.}}{\omega_{\mu\nu}}\overset{\text{antisym.}}{M^{\mu\nu}}} = \Lambda(f)$$

if these $M^{\mu\nu}$ obey certain commutation relations. In spinor space, the unique objects that obey these commutation relations are the $M^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$.

Apply to GR:

If a vector e_μ is infinitesimally parallel transported in the direction of e_a then it obtains an infinitesimal "rotation", namely the infinitesimal Lorentz transformation

$$\omega^\mu{}_\nu(e_a)$$

which is the value of the connection 1-form, i.e.:

$$e_\mu \to e_\mu + \overbrace{\omega_\mu^{\ \nu}(e_a)}^{\text{local value of the connection form}} e_\nu$$

→ From this one can immediately read off again the covariant derivative for vectors:

$$\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a)\, e_\nu$$

- Now, when a spinor s_i is infinitesimally parallel transported in the direction of e_a then it too experiences the infinitesimal rotation, i.e., the infinitesimal Lorentz transformation

$$\omega^\mu{}_\nu(e_a)$$

which is the value of the connection 1 form. Thus:

$$s_i \to s_i - \frac{1}{4}\overbrace{\omega(e_a)_\mu{}^\nu}^{\substack{\text{local infinitesimal Lorentz transformation,}\\ \text{i.e., local value of the connection 1-form.}}} [\gamma^\mu, \gamma_\nu]\, s_i$$

- Since, under infinitesimal parallel transport:

$$s_i \to s_i + \underbrace{\nabla_{e_a}}_{\text{to be determined}} s_i$$

$\Rightarrow$ The covariant derivative of the basis vectors $\{s_i\}$ of Dirac spinors is:

$$\nabla_{e_a} s_i = -\frac{1}{4}\,\omega_\mu{}^\nu(e_a)\,[\gamma^\mu, \gamma_\nu]\, s_i$$

$\Rightarrow$ For general Dirac spinors $\psi(x) = \psi^i(x) s_i$ the Leibniz rule for ∇ yields:

scalar coefficient functions

$$\nabla_{e_a}\psi = \nabla_{e_a}\big(\psi^i(x) s_i\big) = \big(\nabla_{e_a}\psi^i(x)\big)\, s_i + \psi^i(x)\,\nabla_{e_a} s_i$$

i.e.:

$$\boxed{\nabla_{e_a}\psi = e_a(\psi) - \frac{1}{4}\,\omega(e_a)^\nu{}_\mu\,[\gamma^\mu, \gamma_\nu]\,\psi}$$

$e_a(\psi) = s_i\, e_a(\psi^i)$ function, vector field

Dirac equation:

The general relativistic Dirac equation

$$\boxed{(i\gamma^\mu \nabla_{e_\mu} - m)\,\psi = 0}$$

now takes this explicit form:

$$i\gamma^\mu e_\mu(\psi) - i\frac{1}{4}\,\omega(e_\mu)^\nu{}_\rho\, \gamma^\mu [\gamma^\rho, \gamma_\nu]\,\psi - m\psi = 0$$

in a chart, this becomes a directional derivative of ψ.

Remark: The relationship between the Dirac operator $D = i\gamma^\mu \nabla_{e_\mu}$ and the Laplace or d'Alembert operator $\Box$ also becomes:

$$D = d + \delta\,.$$

To this end, one re-interprets the Grassmann algebra of differential forms as a so-called *Clifford algebra*.