Note Title

Recall: If we choose the bases $\{\frac{\partial}{\partial x^\mu}\}$, $\{dx^\nu\}$, then:

Eg: $L_{EM} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$

$$S'[g_{\mu\nu},\psi] = \int\left(\frac{1}{16\pi G} R(g_{\mu\nu}(x)) + L_{matter}\left(g_{\mu\nu}(x), \psi^{(i)}(x), \psi^{(i)}_{,\mu}(x)\right)\right)\sqrt{g}\, d^4x$$

$\frac{\delta S'}{\delta \psi^{(i)}} = 0 \Rightarrow$ Eqns. of motion of matter (Maxwell, Klein Gordon eqns. etc)

$\frac{\delta S'}{\delta g_{\mu\nu}} = 0 \Rightarrow$ Einstein equation:

$$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}}$$

What is the Einstein equation when using a frame so that

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \ ?$$

Recall:

- **Frames $\{\theta^\mu\}$, $\{e_\mu\}$:**

Often, one uses as the bases of $T_p(M)$, and $T_p(M)'$ the canonical bases $\{dx^\mu\}$ and $\{\frac{\partial}{\partial x^\mu}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0,\dots,x^3)$. Thus, when changing coordinate system, $x \to \tilde{x}$, one also usually automatically changes basis in $T_p(M)$, $T_p(M)'$.

Important: The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-) tangent spaces, namely from one canonical basis to another canonical basis, when we change coord. system.

Recall:
(a fixed vector has different coefficients in different bases:
$\xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \frac{\partial \bar{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \bar{x}^\nu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu} \Rightarrow \bar{\xi}^\nu = \frac{\partial \bar{x}^\nu}{\partial x^\mu} \xi^\mu$)

$$\longrightarrow \quad \xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu}$$

We notice: If we choose a fixed basis, say $\{\theta^\mu\}, \{e_\mu\}$ then the coefficients of tensors no longer depend on the choice of coordinates!

E.g.: $\xi = \underset{\text{the same numbers in every coordinate system.}}{\tilde{\xi}^\mu}\, e_\mu$

Conversely: Even staying with one coordinate system, we can freely change our choice of basis in the (co-) tangent spaces:

$$\theta'^\mu = \underset{\text{scalar functions.}}{A^\mu{}_\nu}\, \theta^\nu$$

$$e'_\mu = (A^{-1})_\mu{}^\nu e_\nu$$

So we have e.g.:

$$\xi = \xi^\mu e_\mu = \xi^\mu A^\nu{}_\mu e'_\nu = \xi'^\nu e'_\nu$$

I.e.: $\xi'^\nu = A^\nu{}_\mu \xi^\mu$

Examples:

- The curvature form: $\Omega'^\mu{}_\nu = A^\mu{}_a (A^{-1})_\nu{}^b \Omega^a{}_b$
- But: the connection form $\omega^\mu{}_\nu(\xi) = \xi^\kappa \Gamma^\mu{}_{\kappa\nu}$ obeys:

$$\omega'^\mu{}_\nu = A^\mu{}_a \omega^a{}_b (A^{-1})_\nu{}^b - (dA)^\mu{}_c (A^{-1})_\nu{}^c$$

How to specify frames?

In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:

$$\theta^i(x) = A^i{}_j(x)\, dx^j$$

(Another possibility: Take n scalar functions $f^{(1)}, \dots, f^{(n)}$ and define $\theta^i := df^{(i)}$. For generic functions these $\{\theta^i\}$ will be linearly independent almost everywhere)

Note: the $A^i{}_j(x)$ change nontrivially when changing the coordinate system!

Our choice now: orthonormal frames, or "Tetrads":

- We say that a frame $\{\theta^\mu\}, \{e_\mu\}$ is orthonormal if in this frame, for all $p \in M$:

$$g(e_\mu, e_\nu) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{\mu,\nu} = \eta_{\mu\nu} \quad \text{i.e. if:} \quad g = -\theta^0 \otimes \theta^0 + \sum_{i=1}^{3} \theta^i \otimes \theta^i$$

- Existence? Always: At each $p \in M$ may choose e.g. $\theta^\mu = dx^\mu$ where dx^μ are canonical ON basis at centre of a geodesic cds.

- Uniqueness?

For a given space-time, (M, g), any ON frame yields a new ON frame by transforming the bases through

$$\theta'^\mu(x) = \Lambda(x)^\mu{}_\nu\, \theta^\nu(x),$$

if the linear maps $\Lambda(x)$ preserve the orthonormality:

$$\eta_{\mu\nu}\, \theta'^\mu \otimes \theta'^\nu = \eta_{ab}\, \theta^a \otimes \theta^b$$

recall: this is the defining equation for Lorentz transformations.

$$\text{i.e. if:} \quad \Lambda^\mu{}_a \Lambda^\nu{}_b\, \eta_{\mu\nu} = \eta_{ab} \qquad (*)$$

$\Rightarrow$ Frames are unique up to local Lorentz transformations.

Re-express the degrees of freedom:

- We used to specify space-times through these data: $(\mathcal{M}, g)$
- Now, let us specify space-times, equivalently, through data $(\mathcal{M}, \{\theta^\mu\})$:

Namely:

Assume the $\{\theta^i\}$ are given w. resp. to a basis $\{dx^\mu\}$. through functions $A^\mu{}_\nu$,

$$\theta^\mu(x) = A^\mu{}_\nu(x)\, dx^\nu$$

so that: $g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \eta_{\mu\nu}$ in the basis $\{\theta^i\}$!

Notice: knowing the $A^\mu{}_\nu(x)$, we can reconstruct $g_{\mu\nu}(x)$ in basis $\{dx^\mu\}$:

We use that the abstract g is the same in every basis:

$$g = \underbrace{\eta_{\mu\nu}}_{\text{because it's tetrad}} \theta^\mu \otimes \theta^\nu = \underbrace{\eta_{\mu\nu} A^\mu{}_a A^\nu{}_b}_{= g_{ab}(x)}\, dx^a \otimes dx^b = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu$$

$$\Rightarrow \quad \boxed{g_{ab}(x) = \eta_{\mu\nu} A^\mu{}_a(x) A^\nu{}_b(x)}$$

$\Rightarrow$ $\{\theta^i(x)\}$ indeed determines $g_{\mu\nu}(x)$:

$\Rightarrow$ The $A^\mu{}_\nu(x)$ carry all physical (here shape) info!

How then does $A^{\mu}{}_{\nu}(x)$ encode $C^{i}{}_{jk}, \omega^{i}{}_{j}, \Omega^{i}{}_{j}$?

□ Start with orthonormal frame: $\boxed{\theta^{i}(x) = A^{i}{}_{j}(x)\, dx^{j}}$ (*)

1.) How do the $A^{i}{}_{j}(x)$ determine the $C^{i}{}_{jk}(x)$?

Recall from lecture 11:

$$d\theta^{i}(x) = -\frac{1}{2} C^{i}{}_{jk}(x)\, \theta^{j}(x) \wedge \theta^{k}(x)$$

Here: $d\theta^{i}(x) = A^{i}{}_{j,k}(x)\, dx^{k} \wedge dx^{j}$ because of (*)

$$= -\frac{1}{2} C^{i}{}_{ab}\, \theta^{a} \wedge \theta^{b} = -\frac{1}{2} C^{i}{}_{ab} A^{a}{}_{k} A^{b}{}_{j}\, dx^{k} \wedge dx^{j}$$

$\Rightarrow$ $$A^{i}{}_{j,k} = -\frac{1}{2} C^{i}{}_{ab} A^{a}{}_{k} A^{b}{}_{j}$$

$\Rightarrow$ $$\boxed{C^{i}{}_{ab}(x) = -2 A^{i}{}_{j,k}(x) \left(A^{-1}(x)\right)^{j}{}_{a} \left(A^{-1}(x)\right)^{k}{}_{b}}$$

2.) The $C^{i}{}_{jk}(x)$ yield the $\Gamma^{i}{}_{jk}(x)$ through:

$$\Gamma^{\ell}{}_{ki} := \frac{1}{2}\left(C^{\ell}{}_{ki} - g_{is}\, g^{\ell j} C^{s}{}_{kj} - g_{ks}\, g^{\ell j} C^{s}{}_{ij}\right)$$ (lecture 11)

$$+ \frac{1}{2}\, g^{\ell j}\left(\cancel{g_{ij,k} + g_{jk,i} - g_{ki,j}}\right)$$ ← These all vanish because $g_{\mu\nu,\alpha} = 0$ now

Notice: This simplifies for orthonormal frames with $g_{\mu\nu}(x) = \eta_{\mu\nu}$!

3.) The $\Gamma^{i}{}_{kj}(x)$ yield the $\omega^{i}{}_{j}(x)$:

$$\omega^{i}{}_{j}(x) := \Gamma^{i}{}_{kj}(x)\, \theta^{k}(x)$$

4.) Recall the 2nd structure equation:

$$\Omega^{i}{}_{j}(x) := d\omega^{i}{}_{j} + \omega^{i}{}_{k} \wedge \omega^{k}{}_{j}$$

$\Rightarrow$ We have: $A^{i}{}_{j} \rightarrow \theta^{i} \rightarrow C^{i}{}_{jk} \rightarrow \Gamma^{i}{}_{jk} \rightarrow \omega^{i}{}_{j} \rightarrow \Omega^{i}{}_{j}$

Recall important identities: (torsionless case)

- Structure eqn. I:

$$\Theta^i = D\theta^i = d\theta^i + \omega^i{}_j \wedge \theta^j = 0$$

(Ordinarily: $\theta^i = dx^i \Rightarrow d\theta^i = 0$ and $\omega^i{}_j \wedge \theta^j = 0$ is $\Gamma^i_{jk} = \Gamma^i_{kj}$)

- Structure eqn II:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

(Recall: $R^i{}_{\dots} = \Gamma^i_{\cdot j,\cdot} + \Gamma^i_{\cdot\cdot,\cdot} + \Gamma\Gamma + \Gamma\Gamma$)

- Bianchi identity I:

$$\Omega^i{}_j \wedge \theta^j = 0$$

- Bianchi identity II:

$$D\Omega^i{}_j = 0$$

(From diffeomorphism invariance)

And, in the case of ON frames:

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

Tetrad formulation of GR:

Consider the action, for now, without cosmological constant and without matter:

$$S_{grav} := \frac{1}{16\pi G} \int_B \underset{\text{0-form}}{R} \sqrt{g}\, d^4x$$

Recall Hodge $*$: If

$$\nu = \frac{1}{p!} \nu_{i_1 \dots i_p} \theta^{i_1} \wedge \dots \wedge \theta^{i_p}$$

then

$$*\nu = \frac{1}{p!} \sqrt{g}\, \underbrace{\varepsilon_{i_1 \dots i_n}}_{= \pm 1,\ \text{totally anti-symmetric}} \nu^{i_1 \dots i_p}\, \theta^{i_{p+1}} \wedge \dots \wedge \theta^{i_n}$$

i.e. $* : \Lambda^p \to \Lambda^{n-p}$

Thus:

$$S_{grav} = \frac{1}{16\pi G} \int_B \underbrace{*R}_{\text{4-form}}$$

Aim now: Re-express S'_{grav} in terms of θ^μ and $\Omega^\mu{}_\nu$.

- **Define:**

"capital η" is a $(0,2)$ tensor-valued 2-form

$$H_{\alpha\beta} := *(\theta^\alpha \wedge \theta^\beta) = \frac{1}{2}\underbrace{\sqrt{g}}_{=1}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\gamma\wedge\theta^\delta$$

$$H_{\alpha\beta\gamma} := *(\theta^\alpha\wedge\theta^\beta\wedge\theta^\gamma) = \frac{1}{2}\underbrace{\sqrt{g}}_{=1}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\delta$$

a $(0,3)$ tensor-valued 1-form.

- **Proposition:**

$$*R = H_{\mu\nu}\wedge\Omega^{\mu\nu}$$

(it is a $(0,0)$ tensor-valued 4-form)

i.e.: $$\boxed{S'_{grav}(\theta^\mu) = \int H_{\mu\nu}\wedge\Omega^{\mu\nu}}$$

- **Proof:**

Use $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\kappa\lambda}\,\theta^\kappa\wedge\theta^\lambda \Rightarrow$

$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{1}{2\cdot 2}\,\varepsilon_{\mu\nu\gamma\delta}\,R^{\mu\nu}{}_{\kappa\lambda}\,\underbrace{\theta^\gamma\wedge\theta^\delta\wedge\theta^\kappa\wedge\theta^\lambda}_{=\,\varepsilon_{\gamma\delta\kappa\lambda}\,\theta^\gamma\otimes\theta^\delta\otimes\theta^\kappa\otimes\theta^\lambda}$$

Use also: $\varepsilon_{\mu\nu\gamma\delta}\,\varepsilon_{\gamma\delta\kappa\lambda} = 2(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}) \Rightarrow$

$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{4}{4} R^{\mu\nu}{}_{\mu\nu}\;\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4 = *R \checkmark$$

(need later for derivation of the Einstein equation)

- **Proposition:** $DH_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^\mu = 0$

- **Proof:** $$DH_{\mu\nu} = D\Big(\overbrace{\tfrac{1}{2}\underbrace{\sqrt{g}}_{=1}\,\varepsilon_{\mu\nu\sigma\tau}}^{\text{constant because ON basis}}\,\theta^\sigma\wedge\theta^\tau\Big) = \frac{1}{2}\,\varepsilon_{\mu\nu\sigma\tau}\big(\overbrace{D\theta^\sigma}^{0}\wedge\theta^\tau + \theta^\sigma\wedge\overbrace{D\theta^\tau}^{0}\big)$$

The main proposition:

variation, not co-derivative

Variation of the action with respect to $\underbrace{\delta\theta^\mu(x)}_{\text{i.e., we vary the } A^\mu{}_\nu(x) \text{ by local Lorentz transformations}}$ yields:

$$\delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + d(\text{something})$$

Stokes: $\int_B df = \int_{\partial B} f$

It implies:

$$16\pi G\, \delta S'_{grav} = \int_B \delta\theta^\mu \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + \int_{\partial B} (\text{something})$$

← require variation to vanish at boundary ∂B, so: $= 0$

Definition: The "energy-momentum 1-form" T_μ is defined as the solution to:

$$\delta S_{matter} =: \int_B \delta\theta^\mu \wedge (*T_\mu)$$

⇒ The equation of motion, i.e., the Einstein equation,

$$\frac{\delta(S'_{grav} + S'_{matter})}{\delta\theta^\mu} = 0$$

becomes:

$$\boxed{-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G\, *T_\mu}$$

Exercise: add the cosmological constant.

(it is a (0,1) tensor-valued 1 form)

Remark: The Einstein form $G_\mu := G_{\mu\nu}\theta^\nu$ obeys

$$*G_\mu = -\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho}$$

$$\Rightarrow \quad \boxed{G_\mu = 8\pi G\, T_\mu}$$

Proof of the main proposition:

$$\delta(*R) = (\delta\theta^\mu)\wedge H_{\mu\nu\rho}\wedge\Omega^{\nu\rho} + d(\text{something})$$

Indeed:

$$\delta(*R) = (\delta H_{\mu\nu})\wedge\Omega^{\mu\nu} + H_{\mu\nu}\wedge\delta\Omega^{\mu\nu}$$

Consider the first term:

$$\delta H_{\mu\nu} = \delta\,\overbrace{\tfrac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\rho\sigma}}^{\text{const.}}\,\theta^\rho\wedge\theta^\sigma$$

$$= (\delta\theta^\mu)\wedge H_{\mu\nu\rho}$$

by definition of $H_{\mu\nu\rho}$ above: $H_{\mu\nu\rho} := \frac{1}{2}\sqrt{g}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\delta$

$$\Rightarrow \quad \delta(*R) = (\delta\theta^\mu)\wedge H_{\mu\nu\rho}\wedge\Omega^{\nu\rho} + \underbrace{H_{\mu\nu}\wedge\delta\Omega^{\mu\nu}}_{\text{examine this term:}}$$

$$\delta\Omega^{\mu\nu} \overset{\text{2nd structure equation}}{=} \delta(d\omega^{\mu\nu} + \omega^\mu{}_\rho\wedge\omega^{\rho\nu})$$

$$= d\delta\omega^{\mu\nu} + (\delta\omega^\mu{}_\rho)\wedge\omega^{\rho\nu} + \omega^\mu{}_\rho\wedge\delta\omega^{\rho\nu}$$

$$\Rightarrow \quad H_{\mu\nu}\wedge\delta\Omega^{\mu\nu} = d(H_{\mu\nu}\wedge\delta\omega^{\mu\nu}) - (dH_{\mu\nu})\wedge\delta\omega^{\mu\nu}$$

$$+ H_{\mu\nu}\wedge\delta\omega^\mu{}_\rho\wedge\omega^{\rho\nu} + H_{\mu\nu}\wedge\omega^\mu{}_\rho\wedge\delta\omega^{\rho\nu}$$

$$\overset{\text{by Def. of } D:}{=} (\delta\omega^{\mu\nu})\wedge\underbrace{DH_{\mu\nu}}_{\substack{\text{recall: } =0\\ \text{by Prop. above.}}} + d(H_{\mu\nu}\wedge\delta\omega^{\mu\nu})$$

$\Rightarrow$ Indeed:

$$\delta(*R) = (\delta\theta^\mu)\wedge H_{\mu\nu\rho}\wedge\Omega^{\nu\rho} + d(H_{\mu\nu}\wedge\delta\omega^{\mu\nu}) \quad ✓$$

General Relativity as a "gauge theory"

Recall:

$$S_{grav}(\theta^\mu) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu} \qquad \text{Einstein action}$$

$$-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G \, *T_\mu \qquad \text{Einstein equation}$$

are now the same in all coordinate systems.

In addition:

They are the same also with any choice of ON bases in the tangent spaces, i.e., we have a local symmetry under:

$$\theta^\mu(x) \longrightarrow \tilde{\theta}^\mu(x) = A^\mu{}_\nu(x)\, \theta^\nu(x)$$

The $A^\mu{}_\nu(x)$ are local Lorentz transformations.

Upshot:
- We can start with any matter theory that is invariant under global Lorentz transformations and, through general relativity, turn it into a theory that is invariant under local Lorentz transformations.
- Thereby:
  Derivatives become covariant derivatives.
  A new field is introduced: gravity's Γ.

⇝ This is analogous to the gauge principle of particle physics:

- A global symmetry is "gauged" to become local.
- Derivatives become covariant derivatives
- A new field is introduced.

The gauge principle:

Action for a Dirac field (electrons, quarks etc):

$$S[\psi] = \int \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \, d^4x$$

It has a global symmetry:

$$\psi(x) \to \tilde{\psi}(x) := e^{i\alpha}\psi(x), \text{ i.e., } \bar{\psi}(x) \to \bar{\tilde{\psi}}(x) = e^{-i\alpha}\bar{\psi}(x)$$

$$\Rightarrow S[\psi] \to S[\tilde{\psi}] = S[\psi]$$

However, no local symmetry:

$$\psi(x) \to \tilde{\psi}(x) := e^{i\alpha(x)}\psi(x) \qquad \bar{\psi}(x) \to \bar{\tilde{\psi}}(x) = e^{-i\alpha(x)}\bar{\psi}(x)$$

$$S[\psi] \to S[\tilde{\psi}] \neq S[\psi] \; !$$

Gauge principle: Introduce a new field $A_\mu(x)$ that transforms so as to absorb the extra term:

$$S[\psi, A] := \int \bar{\psi}(x)\Big(i\gamma^\mu \underbrace{\left(\partial_\mu + iA_\mu(x)\right)}_{\text{"covariant derivative"}} - m\Big)\psi(x)\, d^4x$$

Now under

$$\psi(x) \to \tilde{\psi}(x) := e^{i\alpha(x)}\psi(x)$$

$$A_\mu(x) \to \tilde{A}_\mu(x) := A_\mu(x) - i\,\partial_\mu \alpha(x)$$

the action obeys:

$$S[\psi, A] \to S[\tilde{\psi}, \tilde{A}]$$

$$= \int \bar{\psi}(x) e^{-i\alpha(x)}\left(i\gamma^\mu(\partial_\mu + iA_\mu - i\partial_\mu\alpha - m\right) e^{i\alpha(x)}\psi(x)\, d^4x$$

$$= S[\psi, A]$$

Generalization to Yang-Mills theory

Gauging $\psi(x) \to e^{i\alpha(x)}\psi(x)$ introduced $A_\mu(x)$.

and $A_\mu(x)$ turns out to exist: The EM 4-potential.

We "derived" the electromagnetic force!

Notice: $e^{i\alpha(x)} \in U(1)$

$$U(1) = \{ G \in \mathbb{C} \mid G^+ = G^{-1} \}$$

Now give the Dirac particles an extra index (isospin bundle)

$$S'[\psi] = \int \bar{\psi}_a \left(i\gamma^\mu \delta_{ab} \partial_\mu - m\delta_{ab} \right) \psi_b \, d^4x \qquad \left(\sum_{a,b} \text{implied} \right)$$

It's invariant under:

$$\psi_a(x) \to G_{ab}\, \psi_b(x) \qquad \left(\sum_{b=1}^{N} \text{implied} \right)$$

where $G \in SU(N)$

$$SU(N) = \{ G \in M_n(\mathbb{C}) \mid G^+ = G^{-1},\ \det(G) = 1 \}$$

Now, we gauge, i.e., require invariance under:

$$\psi_a(x) \to G_{ab}(x)\, \psi_b(x) \qquad \text{where } G \in SU(N)$$

$\rightsquigarrow$ Invariance of the action now requires new field $B_\mu(x)$:

$$S'[\psi] = \int \bar{\psi}_a \left(i\gamma^\mu \underbrace{\left(\delta_{ab} \partial_\mu + i B_\mu(x)_r T^r_{ab} \right)}_{\text{"covariant derivative"}} - m\delta_{ab} \right) \psi_b \, d^4x$$

and $B_\mu(x)_r \to \tilde{B}_\mu(x)_r = B_\mu(x)_r + \text{complicated}$

Here: $T^r_{ab} \in su(N)$ are a Lie algebra basis, i.e. they are generators of infinitesimal $SU(N)$ transformations.

Upshot:
- $N=2$ Weak force (though mixed with $N=1$ EM)
- $N=3$ Strong force QCD.

Recall:

$$S_{grav}(\theta^\mu) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu} \qquad \text{Einstein action}$$

$$-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G \ast T_\mu \qquad \text{Einstein equation}$$

are the same also with any choice of ON bases in the tangent spaces, i.e., we have a local symmetry under:

$$\theta^\mu(x) \longrightarrow \tilde{\theta}^\mu(x) = A^\mu{}_\nu(x)\,\theta^\nu(x)$$

The $A^\mu{}_\nu(x)$ are local Lorentz transformations.

Our covariant derivative:

$$\nabla_{e_\mu}\left(v^\nu(x)\,e_\nu\right) = \left(\frac{\partial}{\partial x^\mu} v^\nu(x)\right) e_\nu + v^\nu(x)\,\underbrace{\omega^\rho{}_\nu(e_\mu)}\,e_\rho$$

Do the $\omega^\rho{}_\nu$ indeed generate infinitesimal Lorentz transformations?

Plays rôle of A_μ, B_μ but is now gravity!

→ Interpretation of the connection in ON frames:

Q: The connection 1-forms $\omega^\mu{}_\nu$ are not, we know, tensor-valued 1-forms. Wherein do they take their values?

A: The connection 1-forms take values in the set of infinitesimal Lorentz transformations

Intuition?

The connection yields the change under infinitesimal parallel transport – and parallel transport preserves the metric, i.e. it preserves the lengths of vectors, i.e. the change can only be an infinitesimal "rotation", i.e. an infinitesimal Lorentz transformation.

Recall: "Lorentz transformations $\Lambda^\mu{}_a$" are lin. maps obeying:

$$\Lambda^\mu{}_a \Lambda^\nu{}_b \eta_{\mu\nu} = \eta_{ab}$$

$\Rightarrow$ Infinitesimal Lorentz transformations

$$\Lambda^\mu{}_a = \delta^\mu{}_a + \varepsilon^\mu{}_a \qquad \text{with } \left(\varepsilon^\mu{}_a\right)^2 = 0$$

obey:

$$(\delta^\mu{}_a + \varepsilon^\mu{}_a)(\delta^\nu{}_b + \varepsilon^\nu{}_b)\eta_{\mu\nu} = \eta_{ab}$$

i.e.: $\varepsilon^\mu{}_a \eta_{\mu b} + \varepsilon^\nu{}_b \eta_{a\nu} = 0$

$\Rightarrow$ Infinitesimal Lorentz transformations "ILT" are given by

all $\Lambda^\mu{}_a = \delta^\mu{}_a + \varepsilon^\mu{}_a$ which obey: $\boxed{\varepsilon_{ba} + \varepsilon_{ab} = 0}$

Q: Are connection 1-forms ILT-valued?

Proposition:

In orthonormal frames, the 1-form $\omega_{\mu\nu}$ obeys

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

Recall: Absolute exterior derivative: (an anti-derivation)

$$\underbrace{Dt^{a..b}{}_{c..d}}_{\text{any tensor-valued differential form.}} = dt^{a..b}{}_{c..d} + \underbrace{\omega^a{}_i \wedge t^{i..b}{}_{c..d} + \ldots - \omega^i{}_c \wedge t^{a..b}{}_{i..d}}_{\text{play the rôle of the } \Gamma^a{}_{bc}} - \ldots$$

Thus:

$$0 = \nabla g_{\mu\nu} = \underbrace{Dg_{\mu\nu}}_{(0,2)\text{ tensor-valued 0 form}} = \underbrace{dg_{\mu\nu}}_{=0 \text{ because } g_{\mu\nu} = \eta_{\mu\nu} = \text{const}} - \omega^i{}_\mu \wedge g_{i\nu} - \omega^i{}_\nu \wedge g_{\mu i}$$

can drop the $\wedge$ because g is a 0-form.

Recall that by using a tetrad, we achieved that $g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \eta_{\mu\nu}$ everywhere!

i.e. $0 = \omega_{\nu\mu} + \omega_{\mu\nu}$ ✓