Note Title

Fridmann-Lemaître cosmological solutions

Experimental evidence:

Hubble, Humason 1929

- The universe is and has been expanding.
- It appears to be spatially essentially isotropic and homogeneous on scales larger than a few (3-4) billion light years.

(see e.g. Sloan Digital Sky Survey (SDSS) at www.sdss.org)

Idealizing models:

- Assume perfect spatial isotropy and homogeneity:
- ⇝ "Friedmann & Lemaître" (later Robertson & Walker) spacetimes

Concretely:

We assume we can model spacetime as a manifold (M, g) with:

$$M = I \times \Sigma$$

$$g = -dt^2 + a^2(t)\,\bar{g}$$

In the basis $\{dx^\mu\}$ which comes with the coordinate system.

(we will later use an ON frame so that $g_{\mu\nu} = \eta_{\mu\nu}$)

Here:

- I is an interval, $I \subset \mathbb{R}$, and $t \in I$ is called "cosmic time". $a(t)$ is called the "scale factor".
- $(\Sigma, \bar{g})$ is a fully isotropic & homogeneous Riemannian manifold, i.e., a manifold of constant curvature, providing a 3-dim. surface of homogeneity at each point in cosmic time.

What are the possible Riemannian manifolds of constant curvature?

bar for Riemannian mfld of 3 dim.

- The Riemann tensor $\bar{R}_{ijkl}$ must be expressible in terms of a constant, say K, which fixes the curvature's strength, and the tensorial part can only depend on the metric $\bar{g}$.

$\Rightarrow$ Given the index symmetries of $\underbrace{\bar{R}_{ij}}_{\text{antisym.}}\overbrace{{}_{kl}}^{\text{sym.}}$ it should (and does) take the form:

a constant

$$\bar{R}_{ijkl} = K\left(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}\right) \qquad (*)$$

$$\Rightarrow \bar{R}_{jl} = 2K\bar{g}_{jl}, \quad \bar{R} = 6K$$

$\Rightarrow$ Using a "Triad" $\{\bar{\theta}^i\}$: (ON bases of $T_p(\Sigma)$, $\forall p$)

curvature 2-form on Σ

$$\boxed{\bar{\Omega}_{ij} \overset{\text{by}}{\underset{\text{Def.}}{=}} \frac{1}{2}\bar{R}_{ijkl}\,\bar{\theta}^k \wedge \bar{\theta}^l \overset{\text{use}}{\underset{(*)}{=}} K\,\bar{\theta}_i \wedge \bar{\theta}_j}$$

Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called "de Sitter universes".

Rôle of the signature of K:

$K > 0$: $\Rightarrow$ Σ is a 3-dim. sphere (that can be embedded e.g. in a 4 dim euclidean (i.e. flat) space: closed universe

$K = 0$: $\Rightarrow$ Σ is euclidean $\mathbb{R}^3$. flat, infinite universe

$K < 0$ $\Rightarrow$ Σ is a 3-dim. "pseudo sphere". The constant negative curvature means it is everywhere "like a saddle". These Σ also possess ∞ volume.

- **Note:** $\bar{R}$ and therefore K have units $\frac{1}{(\text{length})^2}$. Thus, by suitable choice of unit of length, we can choose units of length so that: (This is usually done in cosmology)

$$K = -1, 0 \text{ or } 1$$

A tetrad for spacetime: $g = -dt^2 + a^2(t)\,\bar{g}$

- Define a convenient **tetrad**, i.e., ON basis of each $T_p(M)$, :

$$\boxed{\begin{aligned} \theta^0 &:= dt \\ \theta^i &:= a(t)\,\bar{\theta}^i \end{aligned}}$$

with t = cosmic time of above

with $\bar{\theta}^i$ being the triad of Σ

- Note: The $\bar{\theta}^i$ were chosen ON with respect to $\bar{g}$.
  The θ^i are ON with respect to g.

We then have, e.g.:

Recall:
The Cartan structure equations express the torsion and curvature forms in terms of the connection form.
(2^{nd} eqn: $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$)

* 1st structure equation on Σ: $(i,j = 1,2,3)$

$$d\bar{\theta}^i + \bar{\omega}^i{}_j \wedge \bar{\theta}^j = 0 \qquad (\Sigma 1)$$

* 1st structure equation on M: $(\mu,\nu = 0,1,2,3)$

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0 \qquad (M1)$$

Determine the 4-connection $\omega^\mu{}_\nu$: (in spatially isotropic & homogenous case)

Strategy: Calculate $d\theta^i$ in two ways:

1.) $d\theta^i = d(a\bar{\theta}^i) = (da)\wedge\bar{\theta}^i + \underbrace{a\,d\bar{\theta}^i}_{\text{use Eq. }\Sigma 1}$

$$= \left(\frac{da}{dt}dt\right)\wedge\bar{\theta}^i - a\,\bar{\omega}^i{}_j \wedge \bar{\theta}^j$$

(use $a\bar{\theta}^i = \theta^i$) $\Rightarrow$

$$= \dot{a}\,\overset{dt}{\overset{\shortparallel}{\theta^0}} \wedge \bar{\theta}^i - \bar{\omega}^i{}_j \wedge \theta^j$$

(use $\bar{\theta}^i = \frac{1}{a}\theta^i$ and $\theta^i \wedge \theta^0 = -\theta^0 \wedge \theta^i$) $\Rightarrow$

$$= -\frac{\dot{a}}{a}\theta^i \wedge \theta^0 - \bar{\omega}^i{}_j \wedge \theta^j \qquad (A)$$

2.) $$d\theta^i \overset{(M1)}{=} -\omega^i{}_\nu \wedge \theta^\nu = -\omega^i{}_0 \wedge \theta^0 - \omega^i{}_j \wedge \theta^j \qquad (B)$$

Compare eqns A,B $\Rightarrow$ $\boxed{\omega^i{}_0 = \frac{\dot{a}}{a}\theta^i \quad \text{and} \quad \omega^i{}_j = \bar{\omega}^i{}_j}$ (Box)

(Intuition: expansion is nontrivial affine connection between space and time)

What is $\omega^0{}_0$? Recall:
$dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$
But $dg_{\mu\nu} = 0$ for ON frames.
Thus $\omega_{\mu\nu} = -\omega_{\nu\mu}$ here.
$\Rightarrow \omega_{00} = 0$

The curvature 2-form:

Recall: 2nd structure equations:

(analogous to: $R^{\cdot}_{\dots} = \Gamma + \Gamma + \Gamma\Gamma + \Gamma\Gamma$)

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu \qquad (M2)$$

$$\bar{\Omega}^i{}_j = d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j \qquad (\Sigma 2)$$

for $i,j \in \{1,2,3\}$ (afterwards we will calculate $\Omega^0{}_i, \Omega^i{}_0$)

$$\Rightarrow \quad \Omega^i{}_j \overset{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j \qquad \text{use (Box)} \Rightarrow$$

$$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

$$\overset{\Sigma 2}{=} \bar{\Omega}^i{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

Recall also:

$$\bar{\Omega}^{ij} = K\,\bar{\theta}^i \wedge \bar{\theta}^j = \frac{K}{a^2}\,\theta^i \wedge \theta^j$$

(It was a consequence of spatial isotropy & homogeneity)

$$\Rightarrow \quad \Omega^{ij} = \frac{K}{a^2}\,\theta^i \wedge \theta^j + \frac{\dot a^2}{a^2}\,\theta^i \wedge \theta^j$$

$\leftarrow$ Recall from equations (box): $\omega^0{}_i = \frac{\dot a}{a}\theta^i$, $\omega_{0i} = -\frac{\dot a}{a}\theta^i$, $\omega_{i0} = \frac{\dot a}{a}\theta^i$, $\omega^i{}_0 = \frac{\dot a}{a}\theta^i$

$$\Rightarrow \quad \boxed{\Omega^{ij} = \frac{K + \dot a^2}{a^2}\,\theta^i \wedge \theta^j}$$

Similarly, one calculates: Exercise: check

$$\boxed{\Omega^{0i} = \frac{\ddot a}{a}\,\theta^0 \wedge \theta^i}$$

Calculate the Einstein tensor:

Recall:

$$\Omega_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\varepsilon}\,\theta^\rho \wedge \theta^\varepsilon$$

$\Rightarrow$ We can read off $R_{\mu\nu\rho\varepsilon}$.

⇒ We obtain the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R.

⇒ We obtain the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Result:

$$\boxed{\begin{aligned} G_{00} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) \\ G_{ii} &= -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} \\ G_{\mu\nu} &= 0 \quad \text{for} \quad \mu \neq \nu \end{aligned}}$$

Exercise: verify

i.e., $G_{\mu\nu}$ is diagonal in this frame.

The energy-momentum tensor:

- From $G_{\mu\nu} = 8\pi G \, T_{\mu\nu}$ we obtain that $T_{\mu\nu}$ must also be diagonal.
- Recall the interpretation of the entries of a diagonal $T_{\mu\nu}$ in terms of matter energy density ρ, matter pressure p and cosmological constant Λ at the origin of geodesic coordinates:

(Why this factor here? Because Λ was traditionally put on the LHS, with the curvature)

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix} - \frac{1}{8\pi G} \Lambda \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

⇒ The only nontrivial dynamics of matter is here its equation of state: $\boxed{\rho = \rho(p) \quad \text{or} \quad p = p(\rho)}$!

What kind of matter causes such a $T_{\mu\nu}$?

Proposition:

The $T_{\mu\nu}$ of any F.L. spacetime is always of the form of that of a perfect fluid.

* The matter doesn't have to be fluid - it could also be e.g. a suitable quantum field.

* But the high symmetry of a F.L. spacetime requires that the matter's $T_{\mu\nu}$ matches that of a perfect fluid.

Proof: Consider the 4-vector field dual to θ^0:

$u = \frac{\partial}{\partial t} = e_0$, i.e.: $u = u^\mu e_\mu$ with $u^0 = 1$, $u^i = 0$.

Using u, $T^{\mu\nu}$ takes the form that characterizes a perfect fluid:

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + (p - \bar{\Lambda})\, g^{\mu\nu} \qquad \bar{\Lambda} = \frac{\Lambda}{8\pi G}$$

Q: If the matter is a fluid, what's the vector field u ?

A: We (i.e. our galaxy) are a particle of the fluid and u is our velocity:

Why? u is tangent to timelike geodesics (that stand still in space because $u \perp e_i \ \forall i = 1,2,3$)

$$\nabla_u u = \nabla_{e_0} \underbrace{e_0}_{\text{dual basis}} = \omega^\mu{}_0 (e_0) e_\mu = \frac{\dot{a}}{a} \overbrace{\theta^i(e_0)}^{=0} e_i = 0$$

Recall:

$\omega^0{}_i = \frac{\dot{a}}{a}\theta^i$

$\omega_{0i} = -\frac{\dot{a}}{a}\theta^i$

$\omega_{i0} = \frac{\dot{a}}{a}\theta^i$

$\omega^i{}_0 = \frac{\dot{a}}{a}\theta^i$

$\omega^0{}_0 = 0$

The Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

now consists of:

$G_{00} = 8\pi G T_{00} \Rightarrow$

$$\boxed{3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G \rho + \Lambda} \quad \text{(A)}$$

$\leftarrow$ "Friedmann equation"

$G_{ii} = 8\pi G T_{ii} \Rightarrow$

$$\boxed{-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi G p - \Lambda} \quad \text{(B)}$$

- Notice that Λ contributes
  - positively to the energy but
  - negatively to the pressure.

Observation:

k/a^2 occurs in (A) and (B), i.e., we can eliminate it:

$$-\frac{1}{2}a\left(\text{Eqn}(B) + \frac{1}{3}\text{Eqn}(A)\right) \text{ yields:}$$

$$\ddot{a} = -\frac{1}{2}a\, 8\pi G\left(\frac{\rho}{3} + p\right) - \frac{1}{2}a\Lambda\left(-1 + \frac{1}{3}\right)$$

$$\Rightarrow \quad \boxed{\ddot{a} = -\frac{4\pi G a}{3}(\rho + 3p) + \frac{1}{3}a\Lambda}$$

Thus for all k: For ordinary matter must have deceleration, i.e., $\ddot{a} < 0$, but a positive cosm. constant Λ can make $\ddot{a} > 0$.

Experimental evidence?

- Supernova distance versus brightness data and evidence from cosmic background radiation:

  $\ddot{a} > 0$ now!

$\Rightarrow$ At present, energy is already sufficiently diluted so that Λ dominates over ρ: $\approx 70\%$ Λ and $\approx 30\%$ ρ^{matter} (dark + visible)

Note: p of a gas of galaxies is negligible.
Note: ρ includes dark matter. Visible matter is only $\approx 3\%$.

- In the far future, ρ & p will have diluted $\to 0$, leaving only Λ. Then, the Friedmann eqn reads:

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = \Lambda$$

Or something other than Λ will dominate $T_{\mu\nu}$ then. Experiments indicate that indeed a faster than exponential expansion may be under way. That cannot come from just Λ dominance alone. See essay topic!

Solutions:

$$a(t) = \begin{cases} \cosh\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k = 1 \\ \exp\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k = 0 \\ \sinh\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k = -1 \end{cases}$$

$\Rightarrow$ Exponential expansion is predicted!

General solution strategy with cosm. constant and matter:

- We have 3 unknown functions of time

  $a(t), \rho(t), p(t)$

  and we have 3 equations that they obey:

  Eqs. A, B and an equation of state $p = p(\rho)$ that depends on the "matter":

  $p_\Lambda(\rho) = -\rho_\Lambda$ for pure vacuum energy (e.g., in very early universe)

  $p(\rho) = \frac{1}{3}\rho$ for pure radiation (e.g., in the early universe)

  $p(\rho) = 0$ for pure dust (e.g., middle aged universe before Λ took over)

- **Observation:**

  The Friedmann eqn. (Eqn. A) only contains a, ρ but not p!

Idea:

- Try to express $\underbrace{\rho}$ as a function of a to obtain: $\rho = \rho(a)$.

this models the dilution of energy density

- Using $\rho(a)$, the Friedmann eqn becomes an ordinary differential equation only for $a(t)$ and we are done!

Indeed, a key equation helps us to find $\rho(a)$:

Proposition: The Einstein eqns A, B, i.e., $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, imply:

$$\boxed{\frac{d}{da}(\rho a^3) = -3 p a^2} \qquad (P)$$

Indeed, when the parameter w in $p = w\rho$ is known, (P) yields $\rho(a)$:

- For dust, $p = 0$ $\Rightarrow \rho \sim a^{-3}$
- For radiation, $p = \rho/3$ $\Rightarrow \rho \sim a^{-4}$

$\rightarrow$ ρ of radiation decays quicker than ρ of dust because radiation is not only diluted, its wavelengths are also stretched, which reduces the energy too.

- For pure Λ: $p = -\rho$ $\Rightarrow \rho = \text{const}$ $\longrightarrow$ ρ of vacuum energy does not dilute!

Intuitive meaning of (P)?

- (P) is the GR version of the continuity equation for (i.e., without heat exchange with an environment) $\rightarrow$ adiabatic expansion: $dE = -p\,dV$
- With $V := a^3$, $E := \rho V$ it yields:

$$d(a^3\rho) = -p\,d(a^3) = -3pa^2 da$$

- Thus: $\frac{d}{da}(a^3\rho) = -3pa^2$ which is indeed (P).

Exact proof of proposition (P):

- The Einstein equation $G^{\mu\nu}=8\pi G T^{\mu\nu}$ and $G^{\mu\nu}{}_{;\nu}=0$ imply $T^{\mu\nu}{}_{;\nu}=0$
- Here: $T^{\mu\nu}=(\rho+p)u^\mu u^\nu + p g^{\mu\nu}$
- Thus:

Notation: $\nabla\cdot u := u^\nu{}_{;\nu}$

$$0 = T^{\mu\nu}{}_{;\nu} = \overbrace{(\rho_{;\nu}+p_{;\nu})\,u^\mu u^\nu + (\rho+p)\,u^\mu u^\nu{}_{;\nu}}^{\text{Leibniz rule}} + p_{;\nu}\, g^{\mu\nu} \qquad (g^{\mu\nu}{}_{;\nu}=0)$$

(using $\nabla_{fv}w = f\nabla_v w \Rightarrow u^\nu \rho_{;\nu} = \nabla_u\rho$) $\Rightarrow$

$$= (\nabla_u\rho + \nabla_u p)\,u^\mu + (\rho+p)\,u^\mu\, \nabla\cdot u + p^{;\mu} \qquad \Big| \cdot u_\mu$$

(using $u^\mu u_\mu=-1$) $\Rightarrow$

$$= -\nabla_u\rho - \cancel{\nabla_u p} - (\rho+p)\,\nabla\cdot u + \cancel{\underbrace{u_\mu p^{;\mu}}_{=\nabla_u p}}$$

$$\Rightarrow \quad 0 = \nabla_u\rho + (\rho+p)(\nabla\cdot u) \qquad (X)$$

It remains now to calculate $\nabla\cdot u$:

$$\nabla\cdot \overset{=e_0}{u} = u^\lambda{}_{;\lambda} = \theta^\lambda(\nabla_{e_\lambda}e_0)$$

Recall: $\nabla_\xi e_b = \omega^c{}_b(\xi)\,e_c$

i.e.: $\nabla_{e_a}e_b = \omega^c{}_b(e_a)\,e_c \Rightarrow$

$$= \theta^\lambda\big(\overbrace{\omega^c{}_0(e_\lambda)}^{\text{numbers}}\, e_c\big) = \underset{\text{used } \theta^\lambda(e_c)=\delta^\lambda_c}{\omega^\lambda{}_0(e_\lambda)} = \underset{\text{since } \omega^0{}_0=0}{\omega^i{}_0(e_i)}$$

Recall that $\omega^i{}_0 = \frac{\dot a}{a}\theta^i \Rightarrow$

$$= \frac{\dot a}{a}\,\theta^i(e_i) = 3\frac{\dot a}{a}$$

$\Rightarrow$ Eqn. (X) becomes:

Thus:

$$\underset{\nabla_u\rho =}{\dot\rho} + 3\frac{\dot a}{a}(\rho+p) = 0 \qquad \left(\text{Recall: } u=\frac{d}{dt}\right)$$

$$\dot\rho\,\frac{a^3}{\dot a} + 3(\rho+p)\,a^2 = 0$$

$$\frac{d\rho}{dt}\frac{dt}{da}\,a^3 + 3\rho a^2 = -3pa^2$$

$$\frac{d\rho}{da}\,a^3 + \rho\, 3a^2 = -3pa^2$$

$$\Rightarrow \quad \boxed{\frac{d}{da}(\rho a^3) = -3pa^2}$$

this is Eqn (P) ✓