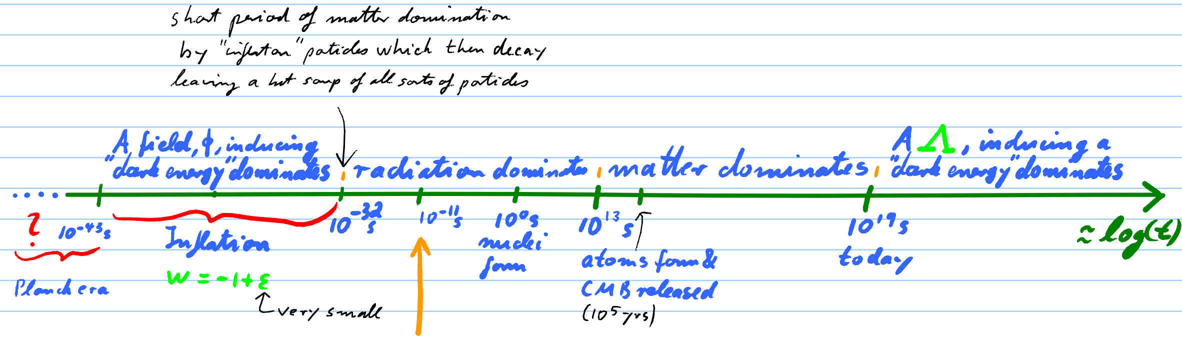
Note Title

Most likely timeline:



at this time, the temperature was so high that particle collisions occurred at a typical energy of $1\,TeV = 1.6 \cdot 10^{-19} \cdot 10^{12}\, J$ which is about the maximal energy that accelerator experiments e.g. at CERN can currently impart on

Evolution of Friedmann-Lemaître spacetimes

- Depending on what is the dominant contributor to $T_{\mu\nu}$, there is an effective "Equation of State": $\boxed{p = p(\rho)}$

- Periods of time in which the eqn. of state can be approximated as:

$$p(\rho) = w\rho \quad \text{with} \quad w = \text{const.}$$

are called Cosmic Epochs.

- For example:

$$\boxed{\begin{array}{ll} \text{Radiation-dominated epoch:} & w = 1/3 \\ \text{Matter ("dust")-dominated epoch:} & w = 0 \\ \text{Dark energy-dominated epoch:} & w = -1 \end{array}}$$

■ For any given epoch, use its $p(\rho)$ to solve (from previous lecture)

Continuity equation $\longrightarrow$ $\frac{d}{da}(\rho a^3) = -3p(\rho)\, a^2$, i.e.: $\frac{d}{da}(\rho(a)\, a^3) = -3a^2 w\, \rho(a)$

to obtain $\rho(a)$, which shows how energy is diluting:

■ **Solution:** $\boxed{\rho(a) = \rho_0\, a^{-3(w+1)}}$ Exercise: verify

■ Key special cases:

$$\rho(a) = \begin{cases} \rho_m\, a^{-3} & \text{in matter-dominated epoch } (w=0) \\ \rho_r\, a^{-4} & \text{in radiation-dominated epoch } (w=1/3) \\ \rho_0\, a^{0} & \text{in dark energy-dominated epoch } (w=-1) \end{cases}$$

dilution of matter, i.e., energy proportional to $\frac{1}{\text{Volume}} \sim \frac{1}{a^3}$

dilution of energy $\sim \frac{1}{\text{Volume}}$ and energy loss due to wavelength stretching $\sim \frac{1}{a}$

vacuum energy due to cosmological constant is of course constant.

We know of no physical mechanism that could cause $w<-1$. Yet, some evidence suggests it might be the case today $\rightarrow$ Note: $w<-1$ would mean $\rho(a) = \rho_s\, a^{\varepsilon}$ ($\varepsilon>0$) i.e. ρ increases with a.

■ Now use $\rho(a)$ to turn the Friedmann eqn. into an ordinary differential equation for $a(t)$:

$$\left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho(a)$$

(we omit the Λ term by agreeing to incorporate Λ in the definition of ρ, p.)

■ Observational evidence: the universe is spatially flat, i.e., $K=0$, in a good approximation.

■ Solution for $K=0$ and $w \neq -1$:

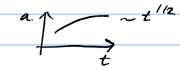
$$\boxed{a(t) = \left(\frac{\pm t}{t_0}\right)^{\frac{2}{3(1+w)}}}$$

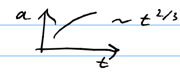
Note that, because $\dot a$ is squared in the Friedmann equation, there is always an expanding along with a contracting solution.

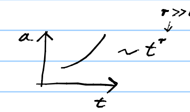
$$a(t) = \left(\frac{\pm t}{t_0}\right)^{\frac{2}{3(1+w)}}$$

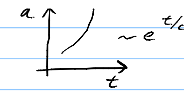
Key epochs: (Exercise/project: what if $K>0$ or $K<0$?)

$$a(t) = \begin{cases} (t/t_r)^{1/2} & \text{in a radiation-dominated epoch } \quad w = 1/3 \\ (t/t_d)^{2/3} & \text{in a matter-dominated epoch: } \quad w = 0 \\ (t/t_p)^{r} & \text{with } r \gg 1 \text{ in a so-called "power law epoch": } \quad w = -1 + \frac{2}{3r} \quad \text{(Exercise: verify)} \\ e^{t/t_v} & \text{in a totally dark energy dominated epoch: } \quad w = -1. \text{ Exercise: Show this.} \end{cases}$$









Definition: Any epoch in which $\ddot{a} > 0$, i.e., in which $w < -1/3$ (exercise: verify), is called an "inflationary epoch".

Most likely timeline:

Best fit today: $K = 0$

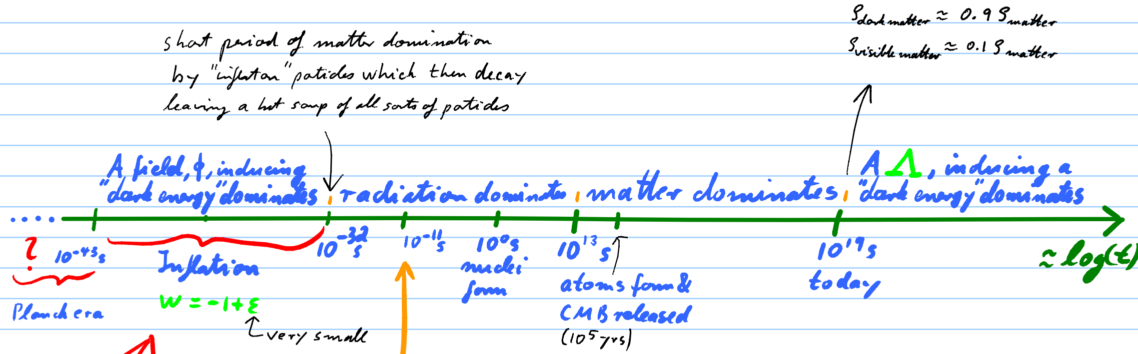
see below for precise definition of $\rho_{critical}$

$\Lambda \approx 0.7\, \rho_{critical}$ ("dark energy")

$\rho_{matter} \approx 0.3\, \rho_{critical}$

$\rho_{dark\ matter} \approx 0.9\, \rho_{matter}$

$\rho_{visible\ matter} \approx 0.1\, \rho_{matter}$



Why inflation?

at this time, the temperature was so high that particle collisions occurred at a typical energy of $1\,\text{TeV} = 1.6\cdot 10^{-19}\cdot 10^{12}\,\text{J}$ which is about the maximal energy that accelerator experiments e.g. at CERN can currently impart on particles.

The flatness problem:

Reconsider the experimental finding of spatial flatness, $K = 0$:

- Rewrite the Friedmann equation

$$3\left(\frac{\dot a}{a}\right)^2 + 3\frac{K}{a^2} = 8\pi G \rho + \Lambda$$

by incorporating Λ in $\rho_{tot} = \rho + \frac{\Lambda}{8\pi G}$ and setting $H := \frac{\dot a}{a}$:

Hubble parameter (const. in space but not in time)

$$H(t)^2 + \frac{K}{a(t)^2} = \frac{8\pi G}{3}\rho_{tot}(t)$$

$\Rightarrow$ At any given time, the critical energy density for $K=0$, i.e., for space to be flat, is:

$$\rho_{crit}(t) = \frac{3}{8\pi G} H(t)^2$$

- How close to critical are we now, and at other times?

Definition:

how close to one is/was it?

$$\Omega(t) := \frac{\rho_{tot}(t)}{\rho_{crit}(t)} \quad \text{, i.e.: } \rho_{tot}(t) = \Omega(t)\frac{3}{8\pi G} H(t)^2$$

- Thus, the Friedmann equation becomes:

$$H(t)^2 + \frac{K}{a(t)^2} = \Omega(t)\, H(t)^2$$

i.e.

$$\boxed{\Omega(t) - 1 = \frac{K}{\dot a(t)^2}}$$

Exercise: check

- Calculate backwards through the matter-dominated epoch, $a \sim t^{2/3}$, $\dot a \sim t^{-1/3}$:

Thus: $\Omega(t) - 1 = K\, t^{2/3}$

(radiation-dominated epoch before: $a \sim t^{1/2} \Rightarrow \dot a \sim t^{-1/2}$
$\Rightarrow \Omega(t) - 1 = K\, t$)

$$\Rightarrow \quad \boxed{\frac{\Omega(t_1)-1}{\Omega(t_0)-1} = \left(\frac{t_1}{t_0}\right)^{2/3}}$$

Notice:
The unit-dependent K dropped out.

- Given that $\Omega(t_1)-1 = \mathcal{O}(1)$ today, at time t_1, much earlier, say at $t_0 = 10^{-6} t_1$, we had

$$\Omega(t_0)-1 = \mathcal{O}(10^{-4})$$

accelerator physics goes so far

At $t_n = 10^{-30} t_1$ (i.e. at $t = 10^{-11}s$) we had

in radiation-dominated epoch the effect is even greater since $\Omega - 1 \sim t$

$$\Omega(t_n)-1 \approx \mathcal{O}(10^{-24})$$

$\Rightarrow$ Flatness is not stable! The universe must have started out flat with tremendous accuracy to be still as flat as we see it today.

Solution to this fine-tuning problem?

- Is there a type of epoch in which the universe evolves towards flatness, rather than away from it?

- Yes! (Brout, Englert, Starobinsky, Linde et al ≈ 1980)
  To this end, conjecture an early cosm. epoch in which

$$\Omega(t)-1 = \frac{K}{\dot{a}(t)^2} \quad \text{with } \dot{a}(t) \text{ increasing with } t.$$

- The current standard model of cosmology therefore postulates an early epoch with:

$$\ddot{a}(t) > 0$$

Recall: We call such an epoch inflationary and it arises whenever $w < -\frac{1}{3}$. ("Inflationary attractor")

- Experimental constraints?

In order to account for the degree of flatness observed today (and cross-checked in the CMB), a period of near-exponential inflation should have expanded the universe by a factor of at least

$$\frac{a(t_{before})}{a(t_{after})} \approx e^{60}$$

The conjecture of an early inflationary epoch also explains

- The absence of exotic high mass particles that would likely have been produced close to Planck time (and only then).

Namely: The inflationary expansion extremely dilutes all particles.

But also: At the end of the inflationary epoch, how did it happen that the universe was filled with a high density of matter?

Currently favored solution:

The inflationary epoch occured when a scalar field ϕ had a large potential:

temporarily large

Recall: $$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

so that, because of the large $V(\phi)$ we had:

$$w = \frac{p_\phi}{\rho_\phi} \simeq -1 \qquad \text{i.e. power law inflation}$$

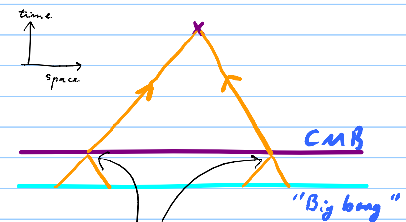
After inflation, $V(\phi)$ becomes the kinetic and mass energy of all sorts of particles, thus making a hot primordial soup. After this "re-heating", followed ordinary big bang cosmology.

Speculation:
How did inflation start?
In any spacetime a single quantum fluctuation of ϕ might elevate $V(\phi)$ locally so as to spawn a new universe!

The conjecture of an early inflationary epoch also solves:

- The "horizon problem":

  Why does the CMB have the same properties even when checking in opposite directions in the sky?



these two areas of the surface that emitted CMB photons do not have a common past. How come they are so similar?

Concretely: Only patches on the CMB sky of angular extent < 1 degree have a common past, if there was no inflation.

- Answer: If the inflationary epoch expanded spacetime sufficiently, (a factor of e^{60} suffices) then all CMB sources have a common past.

The conjecture of an early inflationary epoch also explains

- the occurrence and precise statistics of inhomogeneities in the universe!

- How? The quantum fluctuations of scalar fields (unlike those of spinor fields of, e.g., electrons and vector fields of, e.g., photons) are being amplified in an inflationary epoch, along with those of g.

  $\Rightarrow$ They are thought to have seeded the inhomogeneities in the CMB and therefore ultimately the condensation of hydrogen into galaxies and stars.

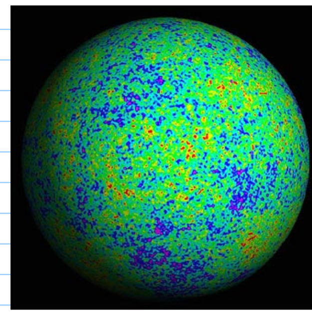
- Experimental check: Statistics from quantum fluctuations matches data with very good precision:

The cosmic microwave background:

- When a hydrogen gas is hotter than $\approx 3000K$ it ionizes, i.e., it is a plasma. The ions interact with light, i.e., the gas is opaque. Below $3000K$ the gas is neutral and therefore transparent.
- The universe, filled with H-gas, made that transition at an age of $\approx 10^5$ys. The light prevalent then is still travelling and arriving from all directions.

Actual temperature data:

($\Delta T \simeq 10^{-6} K$ only!)

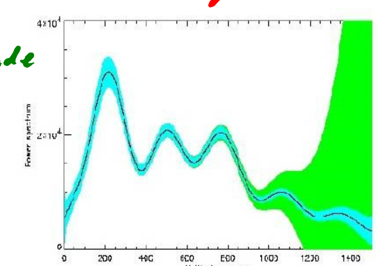


This can be expanded in spherical harmonics $Y_{m,\ell}$, similar to Fourier on a plane:

If caused by quantum fluctuations of ϕ and the metric g, then the predicted statistics was (1980s):

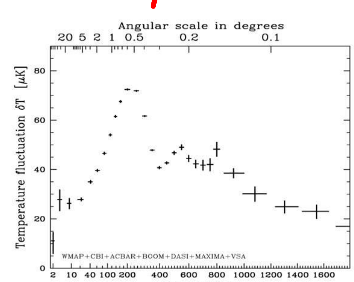
Theory

Amplitude



"angular wavelength^{-1}" ℓ

Experiment



ℓ

Remark: A competing theory held that phase transitions, as the universe cooled, left behind "topological defects" in the vacuum, much like crystal imperfections. Their statistics would be measurably different.

These small inhomogeneities (presumably caused by quantum fluctuations) then explain the statistical distribution of galaxies:

Visualization:

The fluctuations soon grow quickly (Jeans instability) i.e. the evolution of the inhomogeneities then becomes nonlinear and full general relativity, with its nonlinearities, is needed.

Appendix:

The early history of the inhomogeneities is well-described in the linear approximation: (I will here only sketch this. It will be covered in detail in my course next term)

- Start with the action of gravity + inflaton field

$$S = \frac{1}{2}\int\left(-\phi\,\Box\,\phi - V(\phi)\right)\sqrt{g}\,d^4x - \frac{1}{16\pi G}\int R\sqrt{g}\,d^4x$$

- Introduce convenient variables:

In these coordinates, two galaxies always have the same numerical distance

$$y^i := a^{-1}(t)\,x^i \qquad \text{("comoving" coordinates)}$$

$$d\tau := a^{-1}(t)\,dt \qquad \text{("conformal" time)}$$

- Allow the field ϕ to fluctuate:

$$\phi(y,\tau) = \underbrace{\phi_0(\tau)}_{\text{homogeneous}} + \underbrace{\delta\phi(y,\tau)}_{\text{small to 1st order}}$$

- Allow also the metric, g, to deviate locally from spatial flatness, to first order.
- **Recall**: Any vector field can be decomposed uniquely into gradient field + curl field.
- **Similarly**: Decompose the metric perturbations:

$$ds^2 = ds_S^2 + ds_V^2 + ds_T^2$$

contain ↑"scalar" ↑"vector" ↑"tensor" fluctuations.

$$ds_S^2 = a^2(\tau)\left((1+2A)\,d\tau^2 - 2\partial_i B\, dy^i d\tau - [(1-2A)\delta_{ij} + 2\partial_i\partial_j E]\,dy^i dy^j\right)$$
$$ds_V^2 = a^2(\tau)\left(d\tau^2 + 2V_i\, dx^i d\tau - [\delta_{ij} + W_{i,j} + W_{j,i}]\,dx^i dx^j\right)$$
$$ds_T^2 = a^2(\tau)\left(d\tau^2 - [\delta_{ij} + h_{ij}]\,dx^i dx^j\right)$$

- Here:
  - A, B, E are scalar perturbation functions
  - V_i, W_i are 3-vector perturbation functions
  - h_{ij} is a 3-tensor perturbation function

- **Scalar perturbations:**

The only gauge (i.e. cd.) invariant entity is the intrinsic curvature scalar:

$$R^{(3)} = -\frac{a'}{a}\frac{\delta\phi}{\phi_0'} - A \qquad \left({}' \text{ means } \frac{d}{d\tau}\right)$$

It describes the (scalar) quantum fluctuations-induced CMB spectrum.

- Vector fluctuations turn out not to be amplified by expansion $\Rightarrow$ neglect here.
- Tensor fluctuations yield a Weyl curvature, i.e., grav. waves contribution. (see later lecture)

Q: In the CMB?

A: Yes, should be, as the curl of the polarization field.

$\Rightarrow$ High priority experimental search for this curl, the so-called B-polarization of the CMB.

How to calculate the quantum fluctuations?

- Expand the action to 2nd order in $R^{(3)}$ and h_{ij}

$$\Rightarrow \quad S = \frac{1}{2}\int \frac{a^4}{a'^2}\phi_0'^2\left[(\partial_\tau R^{(3)})^2 - \delta^{ij}R^{(3)}_{,i}R^{(3)}_{,j}\right] d\tau\, d^3y$$

$$- \frac{1}{16\pi G}\int a^2\, \partial_\mu h^i{}_j\, \partial^\mu h_i{}^j\, d\tau\, d^4y$$

- Use quantum field theory on curved space to calculate how the quantum uncertainties in $R^{(3)}$ and in $h^i{}_j$ evolve (and amplify!)

Remark: See e.g. my own reseach on how Planck scale physics could observably affect inflationary scalar and tensor predictions.