Horizons & Singularities

Local causal structure

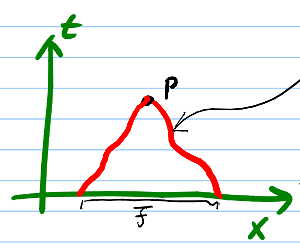
The metric, g, not only defines the "shape" of a pseudo-Riemannian manifold, it also defines what is causal and what is acausal: (by defining what is space-, null- or time-like)

Preparation: • Consider an arbitrary point $p \in M$ and an arbitrary "convex normal neighborhood" of p, i.e., a set $U \subset M$ with $p \in U$ for which holds:

$q, r \in U \Rightarrow$ there exist a unique geodesic connecting q and r.

• **Lemma:** There always exists such a neighborhood.

- Now consider in U:



past light cone is determined by the metric.

some arbitrary coordinate system in which the hyperplanes $t = \text{const.}$ are space-like.

- **Definition:** In order for the laws of matter fields ψ to be called "locally causal" (and therefore reasonable), their equations of motion must allow one to calculate $\psi(p)$ from only the values $\psi(q)$ and finite order derivatives $\psi(q), \ldots$ for all $q \in F$.

Remark:
For massless fields of spin > 1, there is no natural linear equation of motion with such well-defined causality.
Note: Gravitons are spin s=2 but their dynamics is ultimately nonlinear.
(See Wald p.375)

- **Remark:** In Newton's theory these data don't suffice, because there: $c = \infty$

- **Equivalently:** The laws of matter fields are locally causal if signals can be sent between events $q, p \in \mathcal{U}$ only if there is a curve $\gamma \subset \mathcal{U}$ with $\gamma(t_1) = q$, $\gamma(t_2) = p$ whose tangents are non-space-like:

$$g(\dot{\gamma}(t), \dot{\gamma}(t)) \leq 0 \quad \text{for all } t \in [t_a, t_b]$$

Question: Assume that on a differentiable manifold $\mathcal{M}$ only a causal structure is given. To what extent fixes this g?

Answer: Nearly completely!

We will obtain yet another way to describe the "shape" of a curved spacetime!

Theorem:

Assume that on a differentiable manifold $\mathcal{M}$ we don't know the metric, i.e., we can't evaluate

$$g(\xi, \eta)$$

but assume that for all $p \in \mathcal{M}$ and all $\xi \in T_p(\mathcal{M})$ we know for each ξ whether it is space-, light- or time-like, i.e. assume we know:

$\in \{-1, 0, 1\}$

$$\text{sign}(g(\xi, \xi)) \quad \text{for all } p \in \mathcal{M}, \ \xi \in T_p(\mathcal{M})$$

Then, this information already determines the metric tensor up to conformal transformations, i.e., we obtain:

unspecified scalar function: "conformal factor"

$$c(x) \, g_{\mu\nu}(x)$$

also called "holonomic frame"

metric in canonical frame

Remark:

Conformal transformations affect only the length of vectors but leave their mutual "angles" invariant:

$$\cos(\sphericalangle(\xi,\eta)) = \frac{g(\xi,\eta)}{\sqrt{g(\xi,\xi)\,g(\eta,\eta)}} \qquad \left(\frac{c}{\sqrt{c}\sqrt{c}}\right)$$

Proof: ▫ Consider a timelike ξ and a spacelike η.

Are there linear combinations

$$\xi + \lambda\eta$$

that are light-like? If yes, we can assume that we know these λ from knowing the causal structure!

▫ Need to solve this quadratic equation in λ:

$$f(\lambda) = g(\xi+\lambda\eta, \xi+\lambda\eta) = 0 \qquad (*)$$

i.e.: $g^{\mu\nu}(\xi_\mu + \lambda\eta_\mu)(\xi_\nu + \lambda\eta_\nu) = 0$

▫ Eq. $(*)$ has two roots λ_1, λ_2. Are they real?

Yes, because:

ξ timelike $\Rightarrow$ $f(0) < 0$

η spacelike $\Rightarrow$ $f(\lambda) > 0$ for large enough λ

$\Rightarrow$ $f(\lambda) = 0$ has one real root

$\Rightarrow$ Both roots, λ_1, λ_2 of $f(\lambda) = 0$ are real.

▫ Since by assumption we can identify all null vectors we can assume λ_1, λ_2 known.

Lemma:

$$\frac{g(\xi,\xi)}{g(\eta,\eta)} = \lambda_1 \lambda_2$$

Thus, the ratio $\frac{g(\xi,\xi)}{g(\eta,\eta)}$ can be assumed known for all timelike ξ and all spacelike η.

Proof: From $g(\xi+\lambda_{1,2}\eta, \xi+\lambda_{1,2}\eta)=0$

we have: $g(\xi,\xi) + 2\lambda_1 g(\xi,\eta) + \lambda_1^2 g(\eta,\eta) = 0$

and: $g(\xi,\xi) + 2\lambda_2 g(\xi,\eta) + \lambda_2^2 g(\eta,\eta) = 0$

Eliminate $g(\xi,\eta)$ $\Rightarrow$ $\frac{g(\xi,\xi)}{g(\eta,\eta)} = \lambda_1 \lambda_2$ ✓

Exercise: show this

Corollary:

Also the ratios $\frac{g(\xi,\xi)}{g(\xi',\xi')}$ for ξ, ξ' both timelike (or both spacelike) can be assumed known:

$$\frac{g(\xi,\xi)}{g(\eta,\eta)} = \lambda_1\lambda_2 \,;\quad \frac{g(\xi',\xi')}{g(\eta,\eta)} = \lambda_1'\lambda_2' \;\Rightarrow\; \frac{g(\xi',\xi')}{g(\xi,\xi)} = \frac{\lambda_1'\lambda_2'}{\lambda_1\lambda_2}$$

Corollary:

Consider arbitrary non-null vectors α, β.

Then

$$g(\alpha,\beta) = -\frac{1}{2}\left[g(\alpha,\alpha) + g(\beta,\beta) - g(\alpha+\beta,\alpha+\beta)\right]$$

and thus: By lemma, all these ratios can be assumed known

We can consider $g(\xi,\xi)$ to be a fixed, unknown scalar function.

$$\frac{g(\alpha,\beta)}{g(\xi,\xi)} = -\frac{1}{2}\left[\frac{g(\alpha,\alpha)}{g(\xi,\xi)} + \frac{g(\beta,\beta)}{g(\xi,\xi)} - \frac{g(\alpha+\beta,\alpha+\beta)}{g(\xi,\xi)}\right]$$

▫ **Conclusion:**

Therefore, if it is known which vectors are timelike, spacelike or null, then it is possible to calculate

$$g(\alpha,\beta) \text{ at all } p \in M \text{ for all } \alpha,\beta \in T_p(M)$$

up to a scalar prefactor. $\Rightarrow$ Proof of Theorem complete.

▫ **Interpretation:**

The causal structure alone already determines:

- the "angles" between vectors precisely
- the "lengths" of vectors up to a positive scalar function.

▫ **An application to QFT:** arxiv: 1510.02725 w. prev. students of this course!

Implications: Spacetimes (M,g) and $(M,\tilde{g})$ for which

$$\tilde{g} = \phi\, g$$

(if $\phi = 0$ then g not invertible; if $\phi < 0$ then change signature)

ϕ: some positive scalar function

possess the same causal structure.

$\Rightarrow$ Spacetimes fall into "conformal equivalence classes" within which the local causal structure is invariant.

$\rightsquigarrow$ This is very useful to help intuition:

Choose a conformally equivalent spacetime, for which space and time are conformally so much squeezed that infinities turn into a finite distance, all while 45° remain 45° degrees b/c conformality.

Application: Penrose diagrams

Example: Consider Minkowski space, (M, g) in spherical coordinates:

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$$
$$= -dt \otimes dt + dr \otimes dr + r^2 (d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi)$$

with $-\infty < t < \infty, \quad 0 \le r < \infty, \quad 0 \le \phi < 2\pi, \quad 0 \le \theta < \pi$

Now consider the spacetime $(\bar{M}, \bar{g})$ given by:

$$\bar{g} = d\bar{t} \otimes d\bar{t} + d\bar{r} \otimes d\bar{r} + \sin^2(\bar{r}^2)(d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi)$$

with $\overbrace{-\pi < \bar{t} + \bar{r} < \pi, -\pi < \bar{t} - \bar{r} < \pi}^{\text{finite!}}, \bar{r} \ge 0, 0 \le \phi < 2\pi, 0 \le \theta < \pi$

The spacetimes $(M, g), (\bar{M}, \bar{g})$ are related by a diffeomorphism $\bar{M} \to M$:

$$t := \frac{1}{2} \tan\left(\frac{1}{2}(\bar{t} + \bar{r})\right) + \frac{1}{2} \tan\left(\frac{1}{2}(\bar{t} - \bar{r})\right)$$
$$r := \frac{1}{2} \tan\left(\frac{1}{2}(\bar{t} + \bar{r})\right) - \frac{1}{2} \tan\left(\frac{1}{2}(\bar{t} - \bar{r})\right)$$

The diffeomorphism is not isometric, but it is conformal:

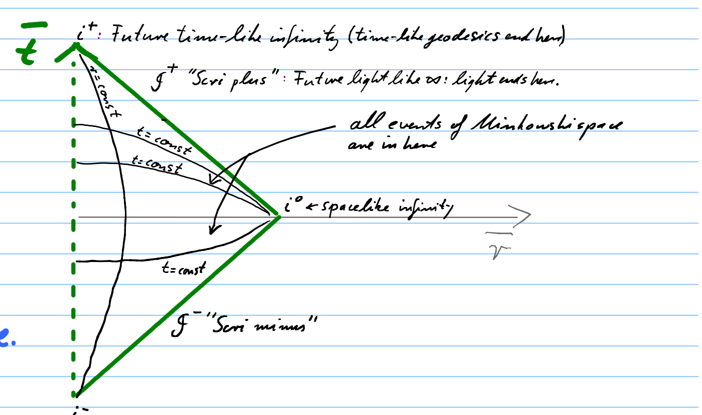
$$g_{\mu\nu} = \phi \, \bar{g}_{\mu\nu} \quad \text{with } \phi = \frac{1}{4} \sec^2\left(\frac{1}{2}(\bar{t} + \bar{r})\right) \sec^2\left(\frac{1}{2}(\bar{t} - \bar{r})\right)$$

Thus, (M, g) and $(\bar{M}, \bar{g})$ have the same causal structure, although $-\pi < \bar{t} + \bar{r} < \pi$ and $-\pi < \bar{t} - \bar{r} < \pi$ and $\bar{r} > 0$.

⇝ Use this to study the causal structure using $(\bar{M}, \bar{g})$ which is of finite size:

Legend:

- Continuous (green) lines: Infinities
- Dotted (green) line: Radius = 0
- Singularities (later): double line.

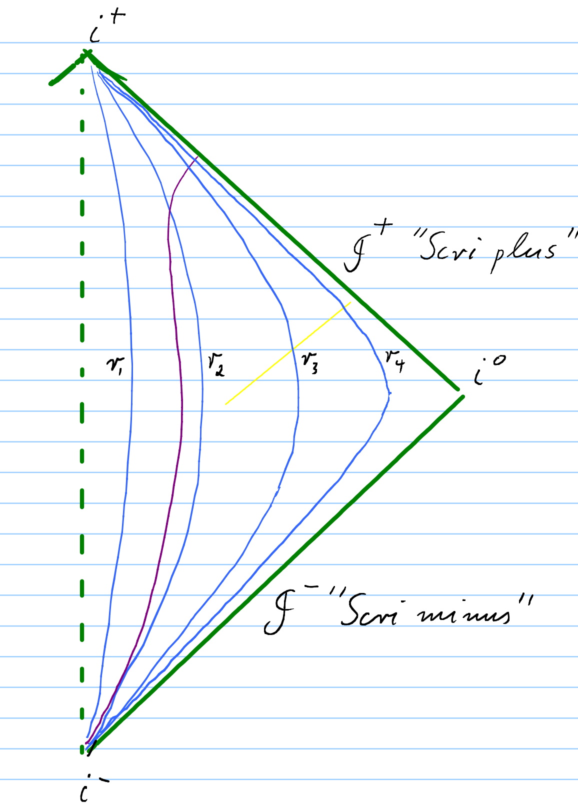


Examples:

A.) geodesic, massive observers, sitting at r_i.

B) same but then uniformly accelerating.

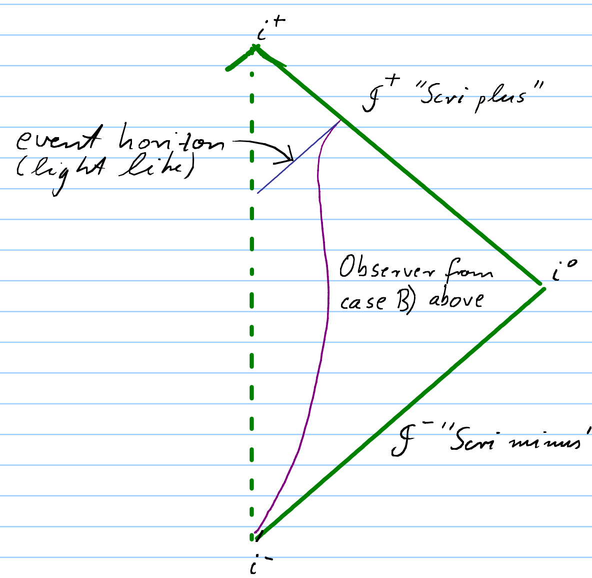
c) light ray



Definition:

An observer's Event horizon (if any) is the boundary of the past of this observer's future causal infinity.

I.e., the event horizon is the boundary of the set of those events that can possibly ever influence the observer, i.e., it's the boundary of the set of events the observer can ever learn about.



Recall FL metric for $K=0$ (i.e. spatially flat $\mathbb{R}^3$)

$$g^{(FL)} = -dt \otimes dt + a^2(t)\, dr \otimes dr + a^2(t)\, r^2 \left(d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi\right)$$

Change to a new time variable τ: "Conformal time"

$$\tau(t) := \int_{t_0}^{t} \frac{1}{a(t')}\, dt'$$

Why useful? Notice: $\frac{d\tau}{dt} = \frac{1}{a(t)} \Rightarrow dt = a\, d\tau$

$$\Rightarrow g^{(FL)} = a^2(\tau)\left(-d\tau \otimes d\tau + dr \otimes dr + r^2\left(d\theta \otimes d\theta + \sin^2\theta\, d\phi \otimes d\phi\right)\right)$$

$$\Rightarrow g^{(FL)} = a^2(\tau)\, \eta$$

$\Rightarrow g^{(FL)}$ is conformally equivalent to Minkowski space!

$\Rightarrow$ Can re-use same Penrose diagram, except range of time τ may be smaller!

Penrose diagrams of F.L. cosmologies: (with $K=0$)

Example: Radiation dominated universe: $a(t) = \sqrt{t}$, $t > 0$

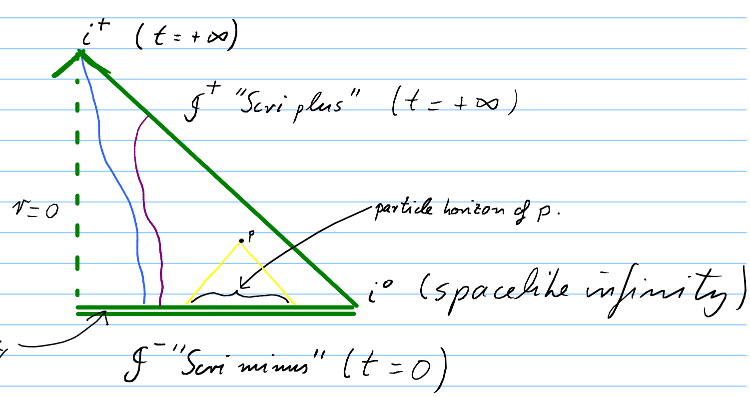
$$\Rightarrow \tau(t) := \int_0^t \frac{1}{\sqrt{t}}\, dt = 2\sqrt{t}\Big|_0^t = 2\sqrt{t} \qquad \boxed{\Rightarrow \tau > 0}$$

$\Rightarrow$ Obtain (τ, r) diagram with $\tau > 0$:

A) geodesic, massive observer

B) same but then uniformly accelerating

C) light rays



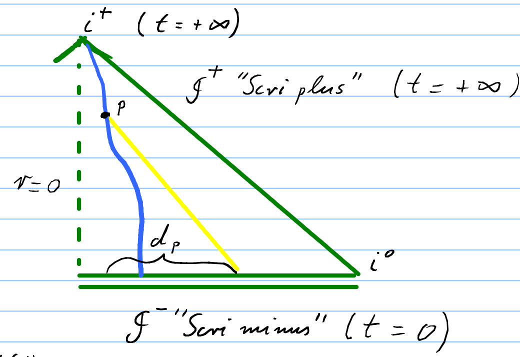
Notice: Singularity at $t=0$ assumed. (Some FL spacetimes are without, e.g. de Sitter: $a(t) = e^{Ht}$)

At finite t, an observer can see only a finite distance.

Def: This distance is called the observer's "Particle Horizon" at time t.

Particle horizon:

How far away, d_p, is the particle horizon at time t?



Recall:

r is the comoving radius, i.e., galaxies sit at fixed (r,θ,φ) at all time. Recall that by definition, $a(t_{today})=1$, i.e., comov. distance = proper distance today.

$$g = -dt\otimes dt + a^2(t)\, dr\otimes dr + r^2(d\theta\otimes d\theta + \sin^2\theta\, d\phi\otimes d\phi)$$

Consider a light ray $\gamma(\lambda) = (\underbrace{\gamma^0(\lambda)}_{=t(\lambda)}, \underbrace{\gamma^1(\lambda)}_{=r(\lambda)}, \underbrace{0}_{=\theta(\lambda)}, \underbrace{0}_{=\phi(\lambda)})$, i.e., emitted radially.

Its tangent is null $g_{\mu\nu}\frac{\partial\gamma^\mu}{\partial\lambda}\frac{\partial\gamma^\nu}{\partial\lambda}=0$, i.e.:

$$\left(\frac{\partial\gamma^0(\lambda)}{\partial\lambda}\right)^2 - a^2(t)\left(\frac{\partial\gamma^1(\lambda)}{\partial\lambda}\right)^2 = 0$$

Note: $\gamma^0(\lambda) = t(\lambda)$

Thus: $\frac{dt}{d\lambda} = \pm a(t)\frac{dr}{d\lambda}$ i.e. $\frac{dr}{dt} = \pm\frac{1}{a(t)}$

Note: this speed is not $=1=c$ because r is the comoving distance. At late times, $a(t)\gg 1$, i.e., $\frac{dr}{dt}$ small, i.e., light crosses comoving distances slowly, - because the same comoving distance becomes larger and larger.

Thus: $d_p = \int_{t=0}^{t}\frac{1}{a(t')}\,dt'$ (It's the comoving distance travelled, and with $a(today)=1$, it's also the current proper distance to what's the furthest we can see.)

For example for us today: $d_p \approx 4\cdot 10^{10}$ light years. (Say since CMB emission)

Recall event horizon:

An observer's event horizon is the boundary of the past of this observer's future infinity.

$\Rightarrow$ If we have a cosmological event horizon, it is the particle horizon that we will have at future infinity.

Q: Do we have a cosmological event horizon?

A: Depends on behavior of $a(t)$ for $t\to\infty$:

I.e., does $d_p^\infty = \int_{t_0}^{\infty}\frac{1}{a(t)}\,dt$ converge to a finite comoving distance?

Recall: $a(t) \sim t^{\frac{2}{3(1+w)}}$

$$\Rightarrow \quad d_p^\infty = \int_0^\infty \frac{1}{a(t)}\,dt \sim \int_0^\infty \overbrace{t^{\frac{-2}{3(1+w)}}}^{t^r}\,dt$$

Notice: convergence iff $r < -1$

$\Rightarrow$ $\exists$ Event horizon iff $w < -\frac{1}{3}$, i.e., if "inflation", i.e., iff $\ddot{a} > 0$!

Notice:

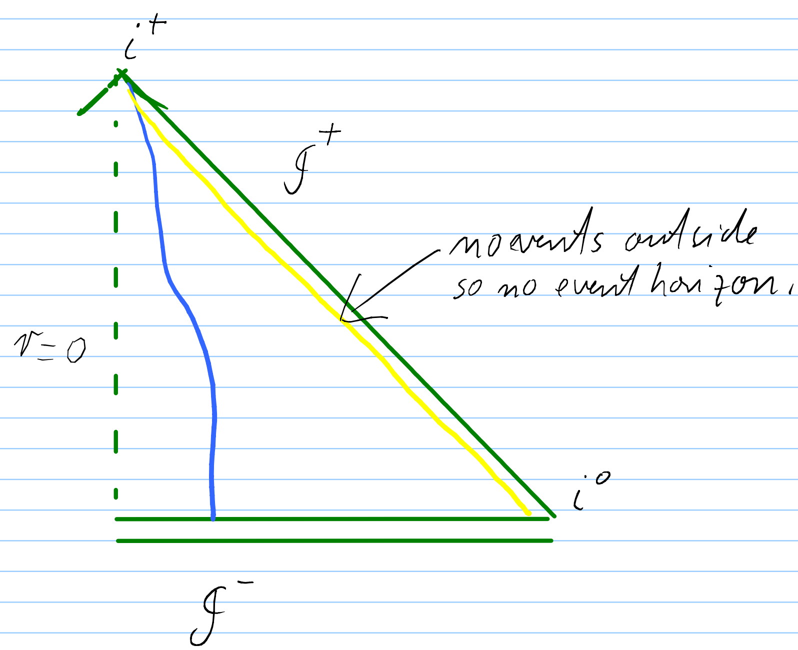
$$d_p(t) = \int_0^t \frac{1}{a(t)}\,dt = \tau(t) = \text{Conformal time !}$$

$\Longrightarrow$ $d_p^\infty = d_p(t=\infty)$ is finite $\Longleftrightarrow$ $\tau(t=\infty)$ is finite

$\Longrightarrow$ Iff inflation then Penrose diagram truncated above at a $\tau = \tau_{max}$ line:

Recall:

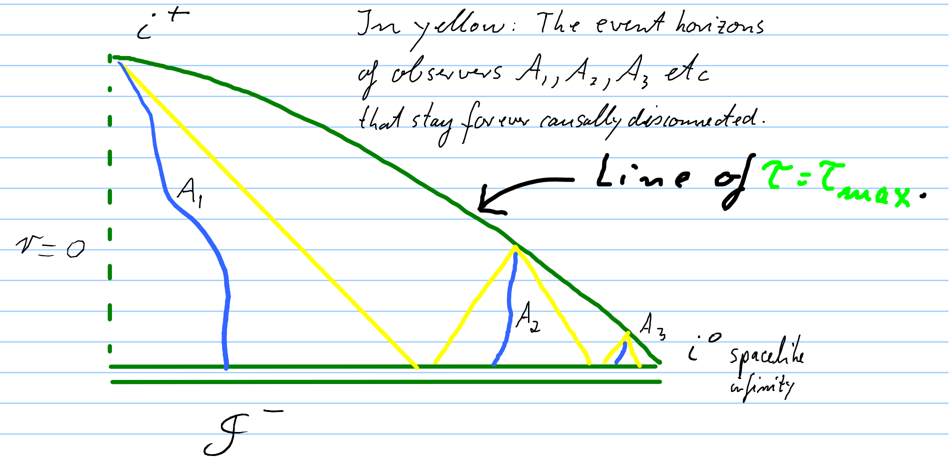
FL spacetime with $K=0$, big bang and no late inflation:



Now with inflation:

(as we have today and presumably in the future)

FL spacetime with $K=0$, big bang and late inflation:



Black holes:

The metric of an eternal, nonrotating, uncharged classical black hole was first found by Schwarzschild in Dec. 1915. It can be written as:

$$ds^2 = \left(1-\frac{r_s}{r}\right)dt^2 - \left(1-\frac{r_s}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

where $r_s = 2GM$ is the Schwartzschild radius.

Notice: At $r = r_s$ this representation of the metric, g, becomes singular. E.g. Kruskal coordinates show that g is regular there.

$\rightsquigarrow$ $r = r_s$ is merely the event horizon (which is light-like!)

Only $r=0$ is a singularity (it is spacelike).

Q: Can we analyze the causal structure using a Penrose diagram, i.e., a conformally equivalent diagram whose light rays are at $\pm 45°$?

Q: I.e., is the metric conformally equivalent to Minkowski space?

Q: Also, can we include the full dynamics of the black hole?

A: Yes, if we consider only the r, t plane. Why?

Theorem: The metric $g_{\mu\nu}$ of any 2-dimensional Lorentzian manifold or sub-manifold reads in suitable coordinates:

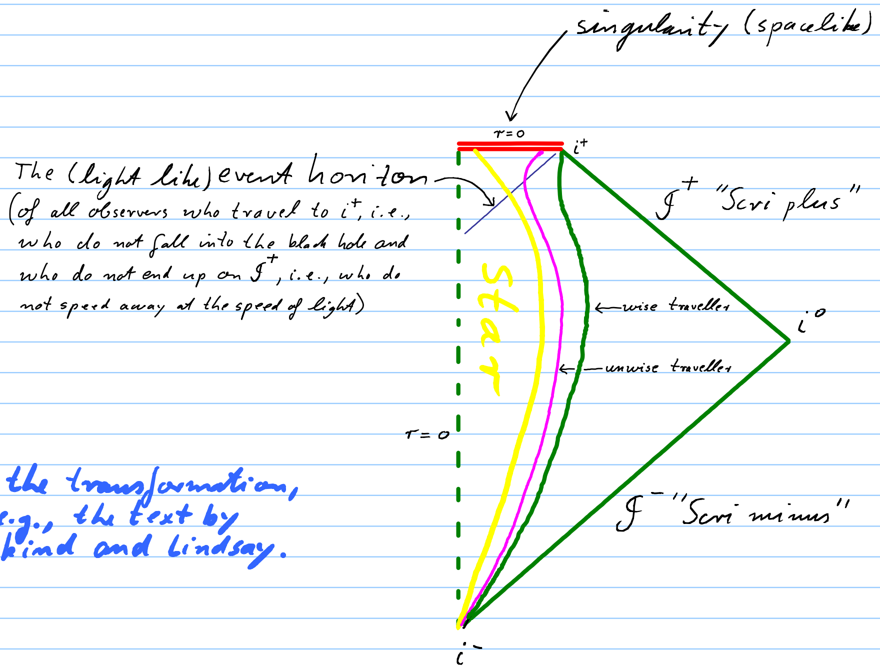
$$g_{\mu\nu}(x) = \lambda(x)\, \eta_{\mu\nu}$$

scalar function

Why? In 2D, $g_{\mu\nu}(x)$ has 3 independent entries and 2 of them can be fixed by choosing the 2 coordinate change functions $\bar{x}_1(x), \bar{x}_2(x)$.

$\Rightarrow$ Every 2D Lorentzian submanifolds of any 3+1 metric has a Penrose diagram.

Example: Collapsing star, forming black hole (non-rotating)



For the transformation, see, e.g., the text by Susskind and Lindsay.

And if the black hole eventually radiates away: