Note Title

Classification of solutions of GR

(and along the way we will introduce some generally useful methods of group theory)

Recall:

- The task is to solve the equations of motion of matter, jointly with the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- In practice, this problem must be simplified, i.e., the number of to-be-determined functions must be reduced.

⇝ Make symmetry assumptions.

Question: How much can we weaken the symmetry assumptions of Friedmann-Lemaître and still get exact solutions?

Strategy:

- Classify cosmological models $(M, g), T_{\mu\nu}$ by the amount and type of symmetry assumed.

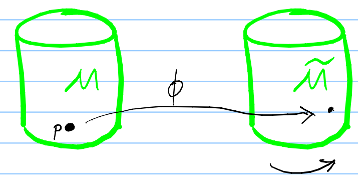
- For each amount and type of symmetry assumed, try to find exact solutions or at least (asymptotic) properties of exact solutions.

Remark: Among the high symmetry models, some come arbitrarily close to F.L. at finite times!

See, e.g., text by Wainwright & Ellis.

Recall: Symmetries & Killing vector fields

- Two spacetimes $(\mathcal{M}, g)$, $(\tilde{\mathcal{M}}, \tilde{g})$ are isometric (and therefore of exactly identical shape) if there is a diffeomorphism $\phi : \mathcal{M} \to \tilde{\mathcal{M}}$ so that the image of the metric g in $\tilde{\mathcal{M}}$ is $\tilde{g}$: $Tg = \tilde{g}$.
- A space-time has a symmetry, if we find such a ϕ for $\tilde{\mathcal{M}} = \mathcal{M}$.
- Example:



ϕ performs a rotation of $\mathcal{M}$ about a symmetry axis, to obtain $\tilde{\mathcal{M}} = \mathcal{M}$ with $Tg = \tilde{g}$.

Note: The set of all symmetries of a manifold $(\mathcal{M}, g)$ forms a "group":

Definition: A "group" G is a set, with an operation, say "$\circ$",

$$\circ : G \times G \to G$$

and a "neutral element", say "e", $e \in G$, such that

$$(a \circ b) \circ c = a \circ (b \circ c) \qquad \forall a, b, c \in G$$

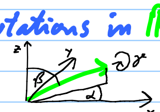
$$a \circ e = e \circ a = a \qquad \forall a \in G$$

$$\exists a^{-1} : \quad a^{-1} \circ a = a \circ a^{-1} = e \qquad \forall a \in G$$

↑ "there exists" ↑ "for all"

Definition: A group G is called a Lie group if G is also a finite-dimensional smooth manifold.

Example: The sets of rotations in $\mathbb{R}^3$ forms a 3-dimensional Lie group, $SO(3)$.
The angles



α, β, γ are coordinates for elements $g \in SO(3)$.

Remarks: ▫ The symmetries of a manifold $(\mathcal{M}, g)$ can be discrete, such as reflections.

▫ But often, the symmetry group of a manifold $(\mathcal{M}, g)$ is actually a Lie group.

Note: ▫ Each $h \in G$ yields an isometric diffeomorphism, by assumption.

$h : \mathcal{M} \to \mathcal{M}$, namely $h : p \to h(p) \quad \forall p \in \mathcal{M}$

▫ Consider the set $\mathcal{O}_p \subset \mathcal{M}$ defined by: $\mathcal{O}_p := \{ q \in \mathcal{M} \mid \exists\, h \in G : h(p) = q \}$

Definition: The set $\mathcal{O}_p$ is called the **Orbit** of p under the action of the group G.

Note: If G is a Lie group then each orbit $\mathcal{O}_p$ is p or a submanifold of $(\mathcal{M}, g)$.

Question: What are the infinitesimal isometric diffeomorphisms?

And what type of mathematical structure do the infinitesimal symmetries form?

▫ **Recall:** The Lie derivative,

$$\begin{aligned} L_\xi Q^{a..b}{}_{c..d} = {} & Q^{a..b}{}_{c..d;k}\, \xi^k \\ & - Q^{k..b}{}_{c..d}\, \xi^a{}_{;k} - \dots - Q^{a..k}{}_{c..d}\, \xi^b{}_{;k} \\ & + Q^{a..b}{}_{k..d}\, \xi^k{}_{;c} + \dots + Q^{a..b}{}_{c..k}\, \xi^k{}_{;d} \end{aligned}$$

yields the rate of change of a tensor Q along the flow of diffeomorphisms ϕ generated by a vector field ξ.

$\rightsquigarrow$ Here, can use L_ξ to differentiate along symmetry group orbits.

▫ Thus, if $\quad L_\xi g_{\mu\nu} = 0$

then ξ generates isometries $\phi : \mathcal{M} \to \mathcal{M}$, $g \to \tilde{g} = g$.

- But $L_\xi g_{\mu\nu} = 0 = \xi^\kappa \underbrace{g_{\mu\nu;\kappa}}^{\substack{\text{always for } \Gamma, g \text{ compatibility} \\ 0}} + g_{\kappa\nu}\xi^\kappa{}_{;\mu} + g_{\mu\kappa}\xi^\kappa{}_{;\nu}$

$\Rightarrow$ A vector field ξ $\overbrace{\text{generates a symmetry}}^{\text{i.e. is itself an infinitesimal symmetry}}$ of spacetime if it is a **Killing vector field**:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \qquad (X)$$

Q: Maximum number, d, of Killing vector fields in n dims?

A: $\boxed{d = n(n+1)/2}$ To see this, note that there are 2 ways to obey Eq. (X):

a) $\xi_{\mu;\nu} = 0 \;\; \forall \nu$, i.e. $\nabla \xi = 0$
Can have maximally n such indep. vectors

b) $\nabla \xi \neq 0$, but then $K_{\mu\nu} := \xi_{\mu;\nu}$ is antisymmetric
Can have at most $n(n-1)/2$ indep. such cases.

$\Rightarrow d = n + n(n-1)/2 = n(n+1)/2$

From a symmetry Lie group to a "symmetry Lie algebra":

General idea:

- A Lie group is a smooth manifold with extra structure: the multiplication.
  Normally the points of a manifold cannot be multiplied!
- Notice: Product of group elements close to $1 \in G$ yields a group element close to 1.
- Consider the tangent space $T_1(G)$ to the point $1 \in G$ of the Lie group manifold G.
- $T_1(G)$ is a vector space and it has extra structure, inherited from the group's multiplication.
- Define the Lie algebra of a group G to be $T_1(G)$, equiped with the inherited "multiplication".
  The identity element of the group, $p = 1$ is also a point of the group's manifold. $T_1(G)$ is the tangent space to this point.

Crucial fact: From knowledge of only the Lie algebra, i.e., only $T_1(G)$ and its "multipliation", the group G can be constructed! (though not always uniquely)

- Let us collect the properties that the inherited multiplications of all Lie algebras share.
- Then, let us define Lie algebras as anything with these properties:

Definition:

A Lie algebra is a vector space A, with an operation $\{,\}$

$$\{,\} : A \times A \to A \qquad \text{"Lie bracket"}$$

obeying $\{r,s\} = -\{s,r\} \quad \forall\, r,s \in A$

and $\{\{r,s\},t\} + \{\{t,r\},s\} + \{\{s,t\},r\} = 0$ "Jacobi identity"

Theorem: Every vector space A with a "multiplication" $\{,\}$ that obeys these axioms is isomorphic to $T_e(G)$ of a Lie group G.

Proposition: The set of Killing vector fields $\xi^{(i)}$ of (M,g) is a Lie algebra.

Exercise: Prove this, i.e., show the following:

Assume $\xi^{(1)}, \xi^{(2)}$ are Killing vector fields of (M,g) and $\alpha, \beta \in \mathbb{R}$.

Then: $\alpha\xi^{(1)} + \beta\xi^{(2)}$ (i.e., they form a vector space)

and $\{\xi^{(1)}, \xi^{(2)}\} := \xi^{(1)}\xi^{(2)} - \xi^{(2)}\xi^{(1)}$

are also Killing vector fields,

and the $\xi^{(i)}$ obey the Jacobi identity.

Summary of the big picture:

1. The symmetries of any (M, g) form a group: they can be concatenated associatively, and all possess an inverse. Recall: there can be discrete symmetries too. Some symmetries are differentiable, parametrized by the flow $\Rightarrow$ the symmetries form a Lie group.

2. Each Killing vector field is the infinitesimal generator of a flow of isometric diffeomorphisms, i.e., of a symmetry.

3. We see here that the Killing vector fields indeed form a Lie algebra.

4. Recall that every Lie algebra generates a Lie group.

Surfaces of homogeneity and the isotropy subgroup:

- Definition:

  Let r be the dimension of the Lie algebra, i.e., also the dimension of the Lie group of symmetries.

- Recall this definition:

  - Consider the set of points $O(p)$ that a point p can flow to along the Killing vector fields.

  - $O(p)$ is called the orbit of $p \in M$ under the action of the symmetry group. We denote the dimension of the orbit by s.

- **Clearly:**

  The dimension of an orbit cannot be larger than the dimension of the symmetry group, i.e.

$$s \leq r,$$

  but $s < r$ easily happens:

- **Example:**
  - Consider $M := \mathbb{R}^2$ and $p = (0,0)$.
  - Then $r = r_{max} = \overset{n=2}{n}(n+1)/2 = \underline{\underline{3}}$ is dim. of sym. group.
  - $\Rightarrow$ The three-dimensional Lie algebra of Killing vector fields is spanned by three Killing vector fields:

- **Concretely:**

$$K^{(1)} := \frac{\partial}{\partial x}, \quad K^{(2)} := \frac{\partial}{\partial y}$$

$$K^{(3)} := y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

  (Group elements generated by them are $e^{a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}}$ and they act as

$$e^{a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}} f(x,y) = f(x+a, y+b)$$

  by Taylor expansion.)

- **Orbit of $p = (0,0)$:**

  $O(p) = \mathbb{R}^2$ because generators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ generate flow to everywhere.

  Def: The surface of homogeneity has dimension $s = 2 < r$

  generated by the Killing vectors (here: $K^{(1)}, K^{(2)}$) which do not have trivial orbits

- **Notice:** Since $n = 2$, at any given point p, only at most 2 Killing vectors can be linearly independent at p.

- Rôle of $K^{(3)}$?

($K^{(3)}$ is the angular momentum and it of course generates rotations: $e^{\alpha K^{(3)}} f(x,y) = f(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$)

The flow generated by $K^{(3)}$ leaves p fixed and rotates everything around p.

- Definition:

We say that those Killing vector fields which do not generate a homogeneity surface, (generalized translations) i.e., which generate a trivial group orbit for a point are generating the isotropy subgroup (generalized rotations) (of the full symmetry group generated by all Killing vectors).

- Dimension, d, of the isotropy subgroup?

Clearly: $\underset{\text{isotropy}}{d} = \underset{\text{full}}{r} - \underset{\text{homogeneous.}}{s}$

Classification of cosmological models

- The classification is with respect to:

  - Dimension of isotropy subgroup d:

(# of conserved 'angular momenta') → $d = 0, 1, 2, 3, 4, 5, 6$

0: anisotropic case

1: at each $p \in M$ one rotational symmetry axis

3: e.g. spatially isotropic case

6: e.g. full Lorentz symmetry

  - Dimension of homogeneity surfaces s:

(# of conserved 'momenta') → $s = 0, 1, 2, 3, 4$

0: inhomogeneous

3: homogeneous on 3 dim orbits

4: homogeneous

- A large body of literature exists on most cases of (d, s):
  - Many exact solutions are known!
  - Many asymptotic behaviors are known!
  - Comprehensive text:

    Wainwright & Ellis, Dyn. systems in cosmology, Cambridge Univ. Press (1997)

- Examples:

s (homogeneity)	d (isotropy)	
4	3	Einstein's static model
4	1	Gödel's model
4	0	Oszvath-Kerr models
3	3	Friedmann Lemaître models
3	1	spatially hom & locally one rot. sym axis
3	0	Bianchi models
⋮	⋮	

Powerful alternative classification approach:

Idea: Classify the possible $T_{\mu\nu}$, then use Einstein equation to obtain classification of curvature.

Proposition:

For every physical energy momentum tensor $T_{\mu\nu}$ there exists a unique timelike vector field u so that $T_{\mu\nu}$ takes this standard form:

$$T_{ab} = \underbrace{\rho}_{\text{scalar}}\, u_a u_b + \underbrace{q_a}_{\text{vector}}\, u_b + q_b u_a + \underbrace{p}_{\text{scalar}}\,(g_{ab} + u_a u_b) + \underbrace{\pi_{ab}}_{\text{tensor}}$$

where q and π are a vector field and a tensor field obeying:

$$q_a u^a = 0, \quad \pi_{ab} u^b = 0, \quad \pi_a{}^a = 0, \quad \pi_{ab} = \pi_{ba}$$

Definition: u is called the "fundamental 4-velocity field"

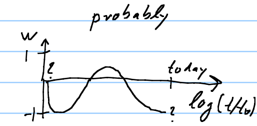
Note: E.g., for a perfect fluid this is the fluid velocity:

$$T_{ab} = \mu u_a u_b + p(g_{ab} + u_a u_b), \quad u_a u^a = -1$$

Recall: equation of state is

$$p = (\overbrace{\gamma - 1}^{w})\mu$$

$$\gamma = \begin{cases} 1 & \text{dust} \\ 4/3 & \text{radiation} \\ 0 & \text{cosmological constant} \end{cases}$$



- **Definition:**

  If (M, g) possesses spacelike $s=3$ homogeneity but the fundamental velocity is not orthogonal to the homogeneity surfaces, then we say that this cosmology is "tilted".

Segré classification:

- A systematic classification of $T_{\mu\nu}$ can be performed, by the analysis of its eigenvalues / eigenvectors. Nontrivial because:
- $T_{\mu\nu}$ is symmetric.
  But, the inner product in the vector space is $g_{\mu\nu} \Rightarrow T_{\mu\nu}$ is generally not hermitean!
- $T^{\mu}{}_{\nu}$ is in a space with the inner product $g^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$, but $T^{\mu}{}_{\nu}$ is generally not symmetric!

Use Jordan normal form:

$\Rightarrow$ Segré classification yields 4 main types of energy momentum tensors $T_{\mu\nu}$.

Recall strategy:

The classification of possible $T_{\mu\nu}$ should, via the Einstein eqns, yield a classification of possible curvatures.

Indeed: In 3+1 dimensions the Einstein equation also reads:

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{{}^{\text{"}}T^{\mu}{}_{\mu}}{T}\right)$$

Exercise: Prove this and notice the dimension-dependence

$\Rightarrow$ The 10 degrees of freedom of $T_{\mu\nu}$ (as a symmetric 4x4 matrix) determine the 10 degrees of freedom of $R_{\mu\nu}$.

$\Rightarrow$ The Segré classification of possible $T_{\mu\nu}$ yields, via the Einstein equation also a classification of possible Ricci tensors $R_{\mu\nu}$.

Q: Does this yield also a classification of the possible Riemann tensors $R^{\mu}{}_{\nu\alpha\beta}$?

A: No! The Ricci tensor contains only 10 of the 20 degrees of freedom of the Riemann tensor! (In 3+1 dim)

Prop.: The information in $R^{\mu}{}_{\nu\alpha\beta}$ is shared among the Ricci tensor $R_{\mu\nu}$ and the so-called Weyl tensor, $C^{\mu}{}_{\nu\alpha\beta}$.

$\Rightarrow$ It remains to classify the possible Weyl tensors!

The Weyl tensor, $C^{am}{}_{sq}$:

$$C^{am}{}_{sq} := R^{am}{}_{sq} - \frac{1}{2}\left(g^a_s R^m_q + g^m_q R^a_s - g^m_s R^a_q - g^a_q R^m_s\right) + \frac{1}{6}\left(g^a_s g^m_q - g^a_q g^m_s\right) R$$

Notice: If $R^a{}_b$ and $C^{am}{}_{sq}$ are given, they determine $R^{am}{}_{sq}$ fully:

$$R^{am}{}_{sq} = C^{am}{}_{sq} + \frac{1}{2}\left(g^a_s R^m_q + g^m_q R^a_s - g^m_s R^a_q - g^a_q R^m_s\right) - \frac{1}{6}\left(g^a_s g^m_q - g^a_q g^m_s\right) R$$

$\rightsquigarrow$ $R^{am}{}_{sq}$ is expressed through $C^{am}{}_{sq}$ and R^{ab}

20 indep. components   10 indep. comp.   10 indep. comp.

$\Rightarrow$ The Weyl tensor $C^{am}{}_{sq}$ indeed contains all that information about the curvature $R^{am}{}_{sq}$ which is not in $R^\mu{}_\nu$.

Determined from $T_{\mu\nu}$ via the Einstein eqn.

$\Rightarrow$ $C^{am}{}_{sq}$ contains all that curvature information which is not determined via the Einstein equation by $T_{\mu\nu}$.

$\Longrightarrow$ $C^{am}{}_{sq}$ describes all that curvature which can exist even where there is no matter! (e.g.: gravity waves)

also e.g. sun's gravity away from the sun in empty space

Proposition

- Assume (M, g) is a 3+1 dimensional Lorentzian manifold.
- Choose any smooth positive scalar function ϕ on M.
- Define $(M, \tilde{g})$ with the new metric $\tilde{g}$ obtained through the "conformal transformation":

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) := \phi(x)\, g_{\mu\nu}(x)$$

Intuition:
Weyl curvature distorts (o o O)
but only Ricci curvature shrinks or expands overall: (o O O)

Then: $\tilde{C}^\mu{}_{\nu\alpha\beta}(x) = C^\mu{}_{\nu\alpha\beta}(x) \quad \forall x \in M$

(Exercise: what would be a proof strategy?)

Historical remark

- Consider the equivalence class of spacetimes $(\mathcal{M}, \tilde{g})$ that are conformally equivalent to Minkowski space:

$$g_{\mu\nu}(x) = \phi^2(x)\,\eta_{\mu\nu}$$

- Einstein and Fokker initially considered a theory in which the metric possesses only this conformal degree of freedom ϕ (to play rôle of Newton's gravitational potential).

Newton gravity does come out correctly as a limiting case!

- Then, $S' = \int_{\mathcal{M}} R\sqrt{g}\,d^4x + \int_{\mathcal{M}} \mathcal{L}_{matter}\sqrt{g}\,d^4x$ and $\frac{\delta S'}{\delta \phi} = 0$ yield:

$$R = 8\pi G\, T^{\mu}{}_{\mu}$$

In electromagnetism $T^{(EM)\mu}{}_{\mu} = 0$ i.e. EM fields would not gravitate.

No gravity waves here because $C^{ab}{}_{cd} = C^{ab\,(\text{Minkowski})}{}_{cd} = 0$

- Equivalence principle ok.
- Light bending & Mercury perihelion shift wrong.

Recall: via the Einstein equation the Segré classification implies a classification of properties of the Ricci tensor $R_{\mu\nu}$.

It remains to classify the Weyl tensor:

Petrov classification:

This is a classification of the Weyl tensor $C^{\mu\nu}{}_{\rho\varepsilon}$, which possesses the 10 remaining degrees of freedom of $R^{\alpha}{}_{\beta\gamma\delta}$.

- $C^{\mu\nu}{}_{\rho\varepsilon}$, just like the Riemann tensor, is antisymmetric in $\mu \leftrightarrow \nu$ and in $\rho \leftrightarrow \varepsilon$, and symmetric in $\mu\nu \leftrightarrow \rho\varepsilon$.

- Thus $C^{\mu\nu}{}_{\delta\epsilon}$ can locally be viewed as a symmetric map from the antisymmetric part $A_p(M)^2$ of $T_p(M)^2$ (so called bi-vectors) into itself:

$$C: A_p(M)^2 \rightarrow A_p(M)^2$$

- But, the inner product in $A_p(M)^2$ is not positive definite!

$\Rightarrow$ C is generally not hermitean.

Therefore, use Jordan normal form again:

Result: 6 main Petrov classes for Weyl curvature: according to eigenvalue/eigenvector decomposition.

Type O: Weyl curvature vanishes

Type D: "Static" Weyl curvature, e.g., in vicinity of a star.

Type N: Transverse gravitational waves, the type LIGO aims to detect. Like light, their strength decays $\sim \frac{1}{r}$ from the source.

Type I: Longitudinal gravitational waves

These waves cause a shear effect. However, they decay fast: $\sim \frac{1}{r^2}$

Why? Gravitational waves, when small enough, travel with speed of light. Like light, they then cannot oscillate longitudinally.

Types II, III: Mixtures of the above.

Potential problem: (with symmetry assumptions):

(E.g. recall that flatness in FL spacetimes is unstable)

- The so-obtained highly symmetric solutions, e.g. Friedman-Lemaître, may possess properties that are peculiar to high symmetry.

(E.g.: In Newtonian gravity, a slightly non-symmetric collapse of a star would not lead to a singularity but to a bounce - think figure skater.)

- E.g.: When a Friedmann-Lemaître solution, or a Schwarzschild solution exhibits a singularity: Is it due to symmetry, or realistic?

- Singularity theorems (see later) confirm the robustness under certain conditions (such as strong energy condition).

→ More confidence in significance of the properties of highly symmetric solutions.