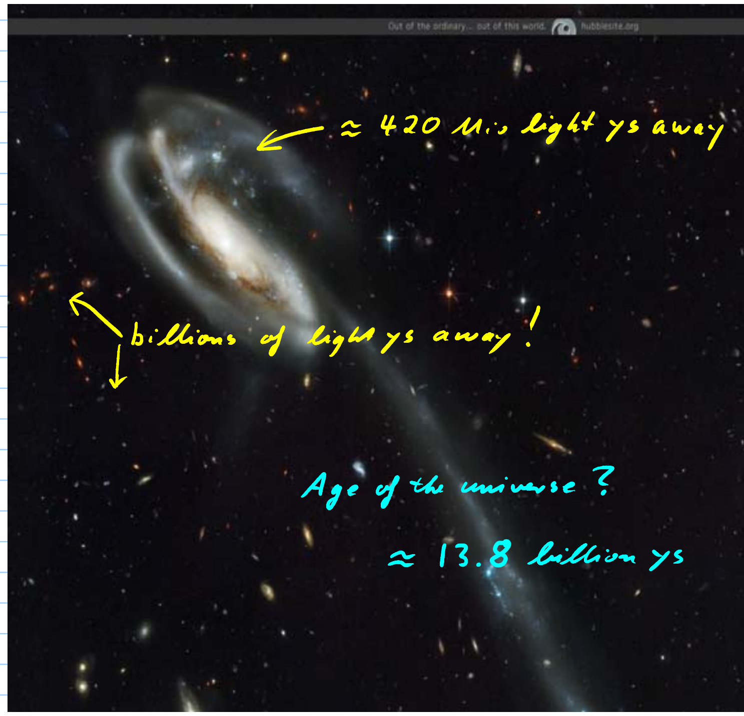
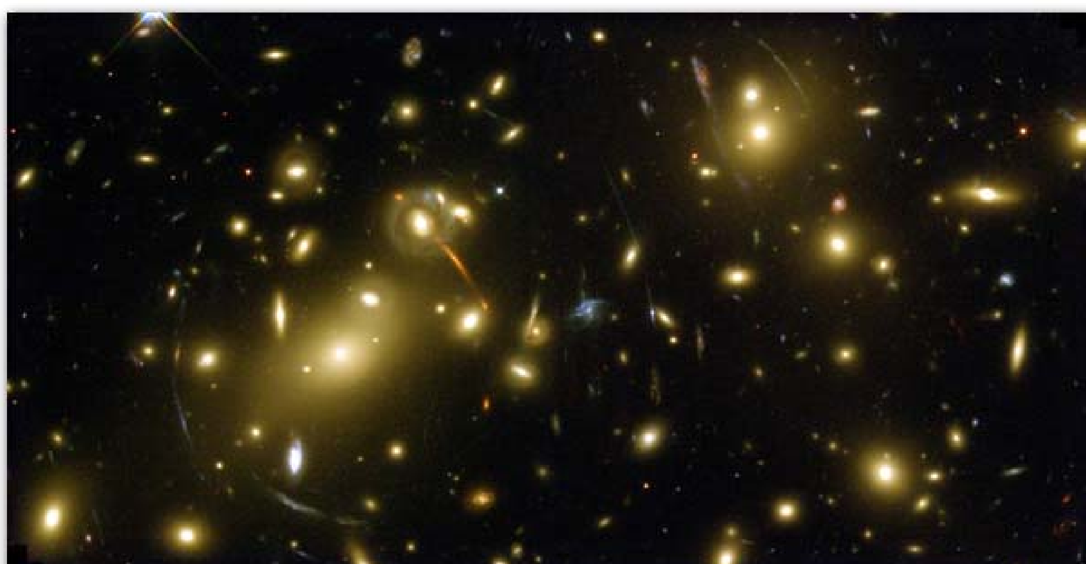
uwaterloo.ca/poi

The 'Tadpole' galaxy



Spacetime's curvature can be seen directly:



HST: ABELL2218

How to describe spacetime?

A. Math

Strategy: ▫ Start with a mere "set" of points (events), M

Then add structure:

▫ Define open neighborhoods (i.e., a "topology" on M)

▫ Define "separability" of points (i.e. Hausdorf condition)

▫ Define "continuity" (preimage of open sets is open)

▫ Define "differentiability" (via chart change diff'ability)

later: ▫ Define tangent & tensor spaces

⋮

$\boxed{\text{Curvature = nontriviality of parallel transport}}$

Other descriptions of curvature?

(Why consider others? May be useful for quantum gravity b/c what's on previous page is likely over idealized.)

▫ Curvature = sum of angles in triangle $\neq \pi$

▫ Curvature = nontriviality of Pythagoras law

▫ Curvature = tidal forces. Math of it: Sectional curvatures

▫ Curvature $\stackrel{?}{=}$ nontrivial sound of object when vibrating

This field is called Spectral Geometry.

Interesting b/c connects mathematical languages of quantum theory (spectra etc) and general relativity.

▫ Curvature $\stackrel{?}{=}$ nontrivial entanglement in vacuum fluctuations

B) Structure and properties of General Relativity?

- Equations of motion

  for scalars, vectors, spinors and curvature

- Symmetries

  local and global conservation laws, if any!

- Tetrad formulation, GR as a gauge theory

- Singularities, and their unavoidability

C) Applications to Cosmology

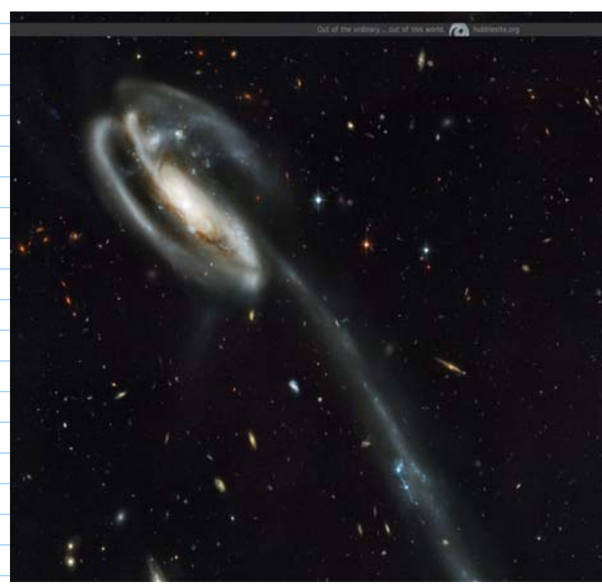
- Classification of exact solutions

- Models of cosmological matter

- FRW models, while using the tetrad formalism to exercise it. (e.g. for later use in quantum gravity)

- Cosmic inflation

- Black holes



A. Pseudo-Riemannian Differential Geometry

▫ Differentiable Manifolds

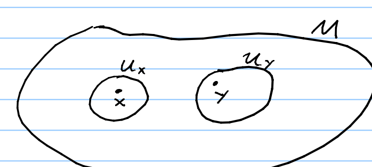
(Riemann ≈ 1850s, Poincaré ≈ 1890s, Whitney ≈ 1930s ...)

Def: An n-dimensional topological Manifold, M, is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Here:

Def: A topological space, M, is a set, together with a specification of subsets U_i, which will be called "open subsets", which must obey $U_i \cap U_j$ is open, and $\bigcup_\lambda U_\lambda$ is open.

Def: A topological space M is called Hausdorff, if it is separable, i.e., if $x, y \in M$ and $x \neq y$ then x, y are elements of some disjoint open sets.



$\forall x, y : x \neq y \ \exists\, U_x, U_y$ open : $x \in U_x, y \in U_y$ and $U_x \cap U_y = \{\}$

↑"for all" ↖"there exist" ↖empty set

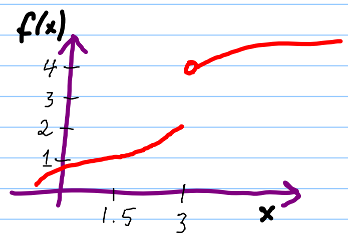
Example: $\mathbb{R}^n$ with its usual definition of open sets.

Now how is the term "homeomorphic" defined?

For this, we need to define "continuity" first:

Recall: If A, B are topol. spaces, then $f: A \to B$ is called continuous if ($\mathcal{U} \subset B$ is open $\Rightarrow$ $\underbrace{f^{-1}(\mathcal{U})}_{= \{x \in A : f(x) \in \mathcal{U}\}} \subset A$ is open)

Example:



$f: \mathbb{R} \to \mathbb{R}$

Choose $\mathcal{U} := (1, 3)$ open

But $f^{-1}(\mathcal{U}) = (1.5, 3]$ not open

Remark: Powerful definition that can be applied very generally.

Why important for us here?

We can now express the idea that a topological Hausdorff space M is continuously parametrizable (as spacetime appears to be)!

Def: Let A, B be topological spaces. Then, a function $f: A \to B$ is called a homeomorphism, if f^{-1} exists and if both f and f^{-1} are continuous.

Def: We say that A is locally homeomorphic to B if for all $p \in A$ there exists an open neighborhood $\mathcal{U}_p$ of p, ($p \in \mathcal{U}_p$) which is homeomorphic to an open set in B.

We choose $B := \mathbb{R}^n$:

Recall:

Def: An n-dimensional topological Manifold, M, is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Now how is the term

Differentiable Manifold defined?

Def: A local homeomorphism,

$$h : \mathcal{U} \to \mathbb{R}^n \ , \quad \mathcal{U} \subset M$$

called "domain"

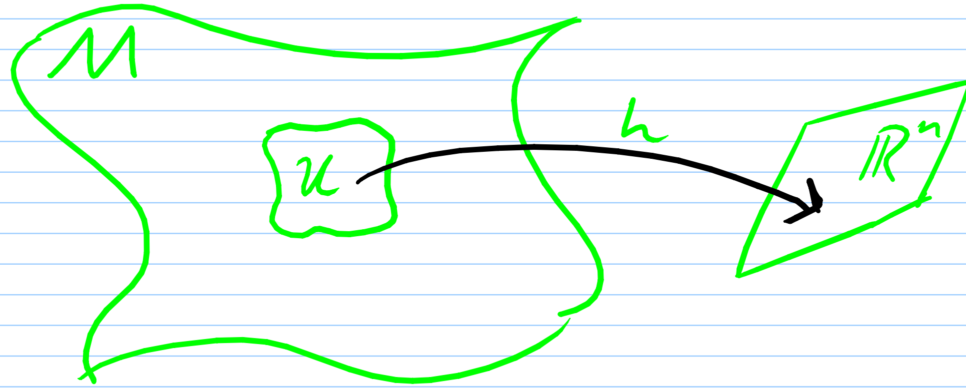
is called a chart of M.

For any point $q \in \mathcal{U}$ its image

$$h(q) \in \mathbb{R}^n$$

is a set of n numbers $(x_1, x_2, \dots, x_n)$ called the coordinates of q.

Def: A chart, h, with domain $\mathcal{U}$,



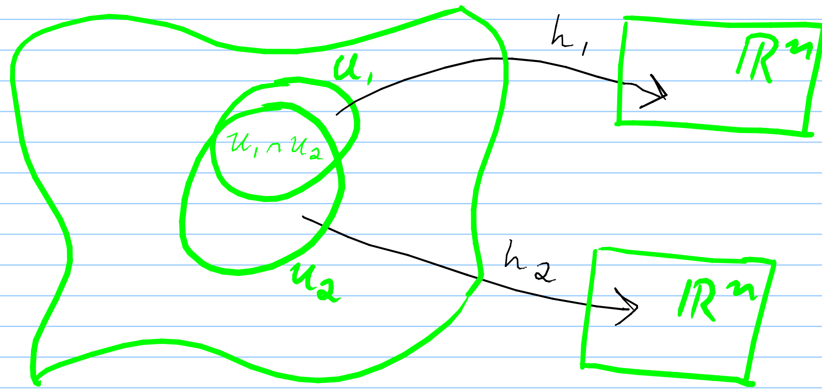
is also called a

local coordinate system for $\mathcal{U}$.

Def: A collection of charts h_α with domains $\mathcal{U}_\alpha$ is called an atlas if $\bigcup_\alpha \mathcal{U}_\alpha = M$.

→ What, if we want to change coordinates, i.e. if we want to re-label the points of (e.g. a subset of) the manifold?

Consider 2 charts h_1, h_2, with intersecting domains $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \phi$:



Then, $h_{12} = h_2 \circ h_1^{-1}$ is a continuous change of coordinates map $h_{12} : \mathbb{R}^n \to \mathbb{R}^n$.

Notice: For maps $\mathbb{R}^n \to \mathbb{R}^n$ we know what differentiability means!

Strategy: Let us define the diffability of an atlas through the diffability of its chart changes:

Def: An atlas is called C^r differentiable, if all its coordinate changes, $h_{\alpha\beta}$, are C^r diffeomorphisms, i.e., r times continuous differentiable.

Strategy: Enlarge atlas so every point of M is in multiple charts. Then, diffability of M is definable through atlas diffability

Def: Given a C^r differentiable atlas, A, we can generate a maximal C^r differentiable atlas, $D(A)$, by adding all charts whose chart changes with charts in A are differentiable.

Def: $D(A)$ is also called a "Differentiable Structure" of class C^r for M.

Def: A differentiable manifold of class C^r is a topol. manifold with a maximal atlas of class C^r, i.e., with a differentiable structure of class C^r.

Theorem: (Whitney)

Every C^k structure with $k \geq 1$ is C^k equivalent to a C^∞ structure (i.e. there is always a suitable set of charts).

I.e. any diffable structure can be smoothened. Any lack of higher diffability is due to unlucky choice of chart.

Def: Since any C^1 manifold is also a C^∞ manifold, we also call diffable manifolds simply smooth manifolds.