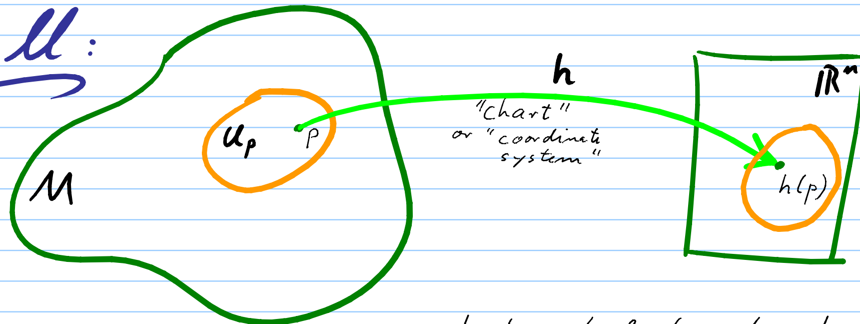
Note Title

GR for Cosmology, Achim Kempf — Lecture 2

Recall:

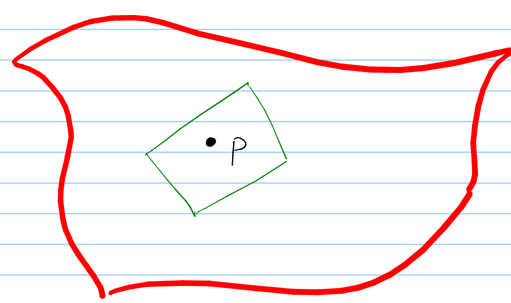


→ charts are tools to get a handle at the otherwise nameless abstract points of the manifold.

Problem:

How to define the abstract "Tangent space, $T_p(M)$," to a diffable mfld at a point p?

Intuition:



⇝ Proper definition should imply:

An n-dim mfld possesses for every point p an n-dim vector space of tangent vectors.

3 equivalent definitions of $T_p(M)$:

1. "Algebraic" definition of $T_p(M)$:

Most powerful b/c no need for coordinates

Idea:
- A tangent vector = directional derivative,
- Derivatives definable through Leibniz rule:

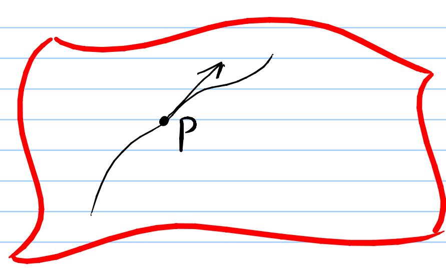
$$(fg)' = f'g + fg'$$

2. "Physicist" definition of $T_p(M)$:

Idea: The elements of $T_p(M)$ are to be vectors ⇒ recognizable by how their components change with charts.

3. "Geometric" definition of $T_p(M)$:

Idea: The elements of $T_p(M)$ are to be actual tangent vectors of one-dim. paths in the manifold, that pass through p.



The 3 defs are equivalent, but:

We'll need all 3 occasionally!

⤳ we will do all 3:

1. Algebraic definition of $T_p(M)$

Idea: a) A tangent vector = directional derivative,

b) Derivatives definable through Leibniz rule:

$$(fg)' = f'g + fg'$$

Key example: $M = \mathbb{R}^n$

a) The tangent vectors ξ at a point p are identified with the directional 1st derivatives:

$$\xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x^i}\Big|_{x=p}$$

b) Thus, tangent vectors at p should be those maps

$$\xi : f \to \xi(f) = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x^i} f(x)\Big|_{x=p}$$

which obey the "Leibniz rule" at p:

$$\text{"}\xi(fg) = \xi(f)g + f\xi(g)\Big|_{\text{at } p}\text{"}$$

Q: How to express the local nature of $\xi \in T_p(M)$ properly?

A: ξ acts on function germs, not on functions.

Def:
- Assume M, N are diffable mflds and $p \in M$.
- We say that two differentiable functions ϕ, ψ are germ-equivalent about p if in a neighborhood $U \subset M$ of p: (i.e. an open set)

$$\phi(q) = \psi(q) \quad \forall q \in U$$

- Each such equivalence class of functions is called a germ at p.
- Then, the "germ" of ϕ at p, denoted Φ_p, is the equivalence class of all functions ψ which are identical to ϕ in some neighborhood of p:

some open neighborhood of p in M.

$$\psi \in \Phi_p \text{ if } \exists\, U_p \;\forall q \in U_p : \quad \phi(q) = \psi(q)$$

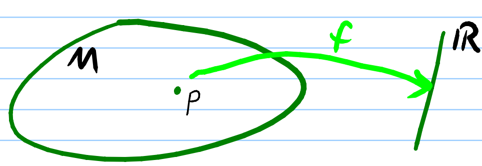
"there exists"

Notice: Assume $\phi: M \to N$ is diffable at $p \in M$.

Then all $\psi \in \Phi_p$ possess the same first derivative at p.

For example:

Consider germs of scalar functions f:



, i.e.: $f: M \to \mathbb{R}$

Note:

- To specify a germ, it suffices to specify any arbitrary one of its functions.
- The set of all germs at p is denoted $\underline{\mathcal{F}(p)}$.

Note: One has for all $c \in \mathbb{R}$ and $f, g \in \mathcal{F}(p)$:

$$\overline{c \cdot f} = c\,\bar{f} \qquad (a)$$

$$\overline{f \cdot g} = \bar{f}\,\bar{g} \qquad (b)$$

$$\overline{f + g} = \bar{f} + \bar{g} \qquad (c)$$

$\Rightarrow$ $\mathcal{F}(p)$ obeys the axioms of an associative algebra.

Finally: Algebraic definition of $T_p(M)$

Recall idea: The elements of $T_p(M)$ are to be 1st derivatives $\Rightarrow$ definable by Leibniz rule.

Definition: The tangent space $T_p(M)$ is the set of "derivations" of $\mathcal{F}(p)$, i.e. the set of linear maps $\xi: \mathcal{F}(p) \to \mathbb{R}$ which obey:

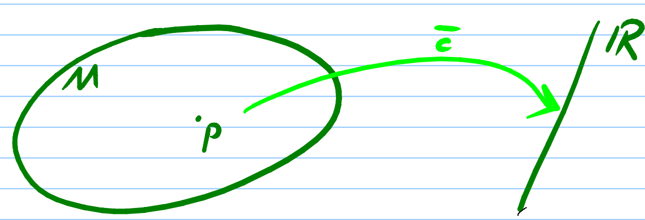
$$\xi\left(\bar{f}_p\,\bar{g}_p\right) = \xi\left(\bar{f}_p\right) \cdot \underbrace{\bar{g}_p(p)}_{= g(p)} + \underbrace{\bar{f}_p(p)}_{= f(p)}\,\xi\left(\bar{g}_p\right)$$

remember this (*)

Remark:

- this definition is abstract enough not only for arbitrary diff'able mflds!
- this definition (as derivations of the algebra of functions) is also suitable for "Noncommutative Geometry": There, (Quantum Gravity) the algebra of functions $F(p)$ is noncommutative.
- Note: Can't do Newton's derivatives then but algebraic def'n of derivation still works.

First example: a constant function, c, and its germ $\bar{c}$.



$c(x) := c$ and c is a constant: $c \in \mathbb{R}$

Then: $\xi(\bar{c}) = 0$ for all $\xi \in T_p(M)$

Proof: $\xi(\bar{c}) = c\,\xi(1) = c\,\xi(1\cdot 1) \overset{\text{Leibniz rule}}{=} c\big(\xi(1)\,1 + 1\,\xi(1)\big)$

$= 2c\,\xi(1) \Longrightarrow \xi(\bar{c}) = 0$ ✓

Example: The case $M = \mathbb{R}^n$

If our definition for $T_p(M)$ is good, we expect that every $\xi \in T_p(M)$ is of the form:

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\bigg|_{x=p}$$

Proof:

- We choose p to have coordinates $x = (0, 0, \ldots)$.
- Assume $\xi \in T_p(M)$ and $\bar{f} \in \mathcal{F}(p)$.

- Notation: $h_{,i}(a^1, \ldots, a^n) := \frac{\partial}{\partial a^i} h(a^1, \ldots, a^n)$

Then:

(Note: these are not 3 numbers! These are 3 function germs, i.e., 3 equivalence classes of functions.)

germ of a constant function

$$\xi(\bar{f}(x)) = \xi\Big(\overline{f(0)} + \overline{f(x)} - \underbrace{\overline{f(0)}}_{\text{a constant function}}\Big)$$

$$\overset{(c)}{=} \xi\left(\overline{f(0)} + \int_0^1 \frac{d}{dt} \bar{f}(tx^1, \ldots, tx^n)\, dt\right)$$

$$\overset{(b)}{=} \underbrace{\xi(\bar{f}(0))}_{=0} + \xi\left(\int_0^1 \sum_{i=1}^{n} \frac{\partial \bar{f}(tx^1, \ldots, tx^n)}{\partial (tx^i)} \frac{d(\overline{tx^i})}{dt}\, dt\right)$$

$$= \xi\left(\int_0^1 \sum_{i=1}^{n} \bar{f}_{,i}(tx^1, \ldots, tx^n)\, \bar{x}^i\, dt\right)$$

Linearity of $\xi \Rightarrow$

$$= \sum_{i=1}^{n} \xi \left(\int_0^1 \bar{f}_{,i}(tx^1, \dots, tx^n) dt \cdot \bar{x}^i \right)$$

Leibniz rule $\Rightarrow$

$$= \sum_{i=1}^{n} \xi \left(\int_0^1 \bar{f}_{,i}(tx^1, \dots, tx^n) dt \right) \cdot \overbrace{\bar{x}^i \Big|_{x=p=0}}^{=0}$$

$$+ \sum_{i=1}^{n} \left(\int_0^1 \bar{f}_{,i}(tx^1, \dots, tx^n) dt \right) \Bigg|_{x=p=0} \cdot \xi(\bar{x}^i)$$

(Remember from (*) above)

$$= \sum_{i=1}^{n} \xi(\bar{x}^i) \int_0^1 \bar{f}_{,i}(0, \dots, 0) dt$$

$\left(\int_0^1 c \, dt = c \Rightarrow \right)$

$$= \sum_{i=1}^{n} \xi(\bar{x}^i) \frac{\partial}{\partial x^i} f(x^1, \dots, x^n) \Big|_{x=p=0}$$

$\Rightarrow$ Indeed, every $\xi \in T_p(M)$ is of the form

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\Big|_{x=p} \qquad (\mathrm{I})$$

namely with

$$\xi^i = \xi(\bar{x}^i) \qquad (\mathrm{II})$$

$\square$

Notice: Knowing how ξ acts on the coordinate functions $\bar{x}^i$ yields ξ^i (from II) and thus it means we know how ξ acts on all functions $\bar{f} \in \mathfrak{F}(p)$, namely through (I).

But:

- This was the simple example:

$$M = \mathbb{R}^n$$

- How does our definition of $T_p(M)$ work for $M \neq \mathbb{R}^n$, concretely?

- **Recall:**

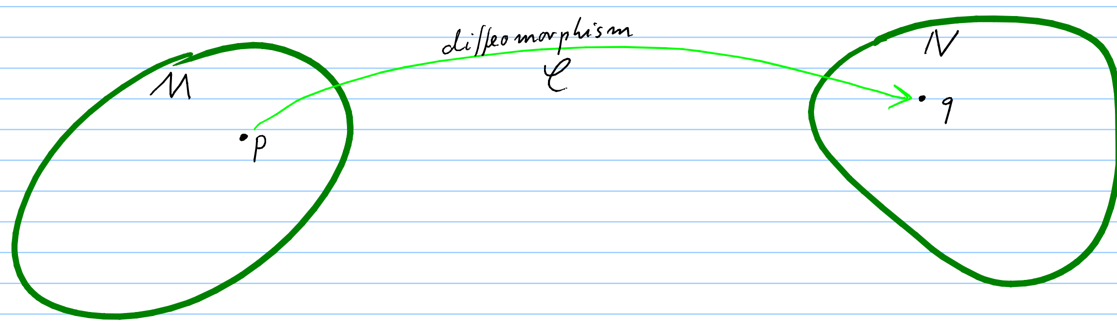
h gives abstract points a name, i.e. makes them concrete.

- **Problem:** How to make abstract $\xi \in T_p(M)$ concrete?

- **Solution:** Make use of charts in clever way!

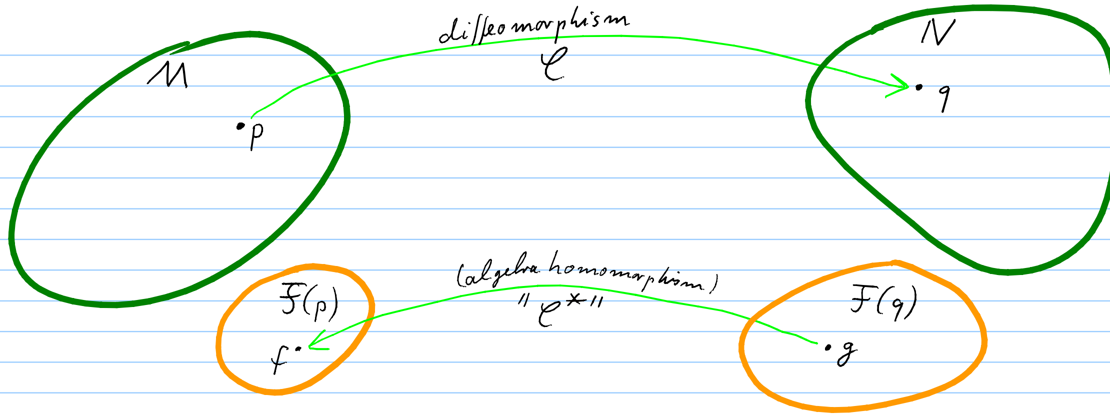
Preparation: $T_p(M)$ and Diffeomorphisms.

Consider two diff'able manifolds, M and N:



Note: If $N = \mathbb{R}^n$, then φ is a chart.

(that's the case we'll need but it's easy to keep a general N too)

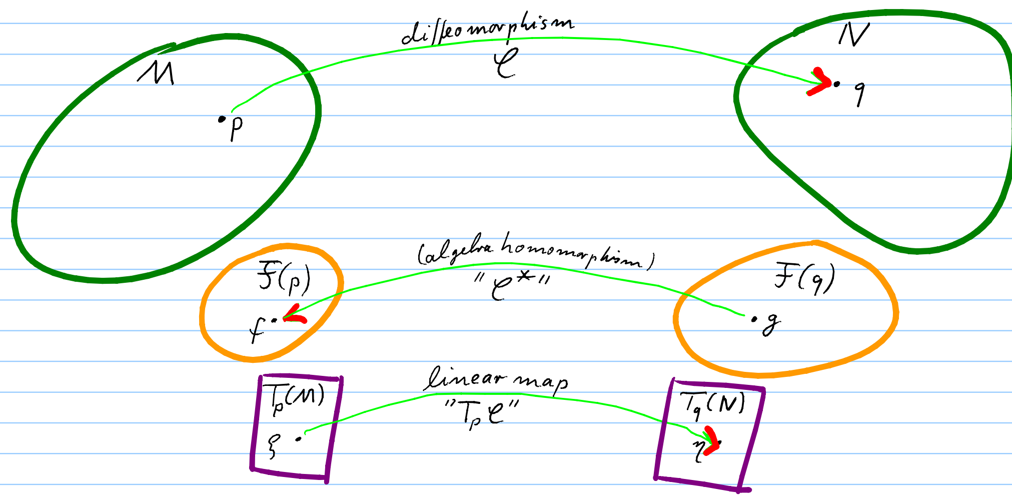


Here: F(q) and F(p) are algebras of function (germs).

Given φ we obtain a map $\varphi^*: \mathcal{F}(q) \to \mathcal{F}(p)$

$$\varphi^*: g \to f = \varphi^*(g) \quad \text{with } f(x) = g(\varphi(x)) \; \forall x \in M$$

i.e.: $f = \varphi^*(g) = g \circ \varphi$ (+)



Here: Given $\varphi^*: \mathcal{F}(q) \to \mathcal{F}(p)$ we obtain the "tangent map":

$$T_p\varphi: T_p(M) \longrightarrow T_q(N)$$

$$T_p\varphi: \xi \longrightarrow \eta$$

(When choosing $M = \mathbb{R}^n$, we obtain the desired concrete representation of $T_p(M)$ this way)

- Namely: $\eta = \xi \circ \varphi^*$

  i.e.: $\eta(g) = \xi(\varphi^*(g))$

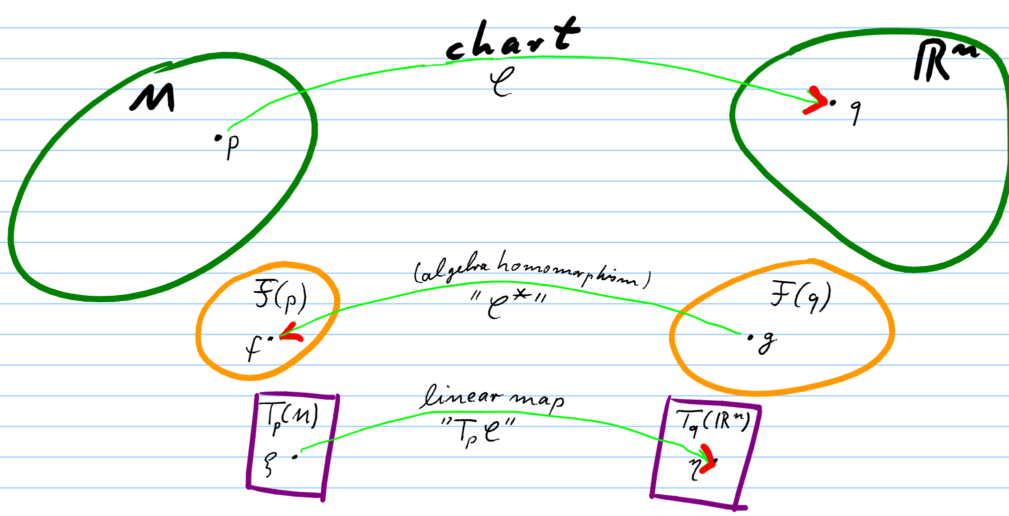
- From (+) $\Rightarrow$

  $\eta(g) = \xi(g \circ \varphi)$

The crucial special case:

- $N = \mathbb{R}^n$ (with $n = \dim(N)$)
- φ is invertible
- ($\Rightarrow$ φ^* is algebra isomorphism)
- $\Rightarrow$ $T_p\varphi$ is vector space isomorphism

$\Rightarrow$ We do obtain a concrete handle on the abstract tangent vectors $\xi \in T_p(M)$, given a chart h:



Namely:

- Given a chart φ, every abstract point $p \in M$ has a concrete image $\varphi(p) \in \mathbb{R}^n$, and:

- Every abstract vector $\xi \in T_p(M)$ has a concrete image $\eta \in T_{\underbrace{\varphi(p)}_{x=q}}(\mathbb{R}^n)$ namely:

$$\eta = T_p\varphi(\xi)$$

- The image η is concrete because η is tangent vector to a point $q \in \mathbb{R}^n$, and it therefore must take the form (we showed this):

$$\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\bigg|_{x=q}$$

concrete numbers.

Conversely: (and very conveniently)

- Assuming a fixed φ, any choice of a $q = (x^1, \ldots, x^n)$ denotes a $p \in M$ and any choice of a $(\eta^1, \ldots, \eta^n)$ denotes a $\xi \in T_p(M)$.

some numbers

□ E.g. $\eta = \frac{\partial}{\partial x^i}\Big|_{x=q}$ is the image of some abstract $\xi \in T_p(M)$, for fixed i.

Notation: $\xi = $ "$\frac{\partial}{\partial x^i}\Big|_{x=p}$"

symbolic notation

Next:

If we hold p and $\xi \in T_p(M)$ fixed, how do the numbers $(x^1, \ldots, x^n)$ and $(\eta^1, \ldots, \eta^n)$ change when we change the chart? ⟶ Physicists' def of $T_p(M)$