Note Title

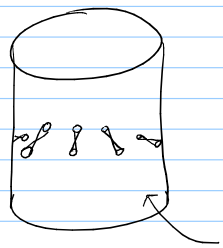
Causal Structure & "Singularities"

Definition:

We say that (M, g) is <u>time-orientable</u> if at each point $p \in M$ we can separate the non-spacelike vectors

$$\xi \in T_p(M), \; g(\xi, \xi) \leq 0$$

into two classes, which will be called future-directed and past-directed so that this is separation is continuous in M.



Consider e.g. such a spacetime, which is the outside of a cylinder.
If the metric has the drawn light cones, M is not (continuously) time-orientable.

Lemma:

Iff (M, g) is time-orientable, then there exist smooth nonvanishing timelike vector fields, v, on M.

Note:

If such a v is given, we can define at each $p \in M$ the future-directed vectors ξ as those non-spacelike vectors for which $g(v, \xi) > 0$, i.e. the past-directed non-spacelike ξ obey $g(v, \xi) < 0$. (Alternatively the "future" and "past" designations can be globally swapped.)

Why? Consider basis in $T_p(M)$ so that $g_{\mu\nu} = \eta_{\mu\nu}$ and $v = \begin{pmatrix} v_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then $g(v, \xi) = -v_0 \xi_0$, i.e. "sign" of ξ is well-defined relative to v.

Assumption: We will henceforth only consider time-orientable space-times, and we will always assume a time orientation has been chosen.

Time-like curves

Definition: A curve (or "path") $\gamma: \underset{\mathbb{R}}{\underset{\cup}{J}} \to M$ is called a

"future-directed timelike curve"

if each $\dot{\gamma}(t) \in T_{\gamma(t)}(M)$ is a future-directed timelike vector.

Note: there is no restriction to geodesics. These are all paths that material objects can travel, given suitable engine or outside forces.

Note: Here $\dot{\gamma}(t) = 0$ is not allowed because timelikeness requires $g(\dot{\gamma}, \dot{\gamma}) < 0$.

Definition: A curve (or "path") $\gamma: \underset{\mathbb{R}}{\underset{\cup}{J}} \to M$ is called a

"future-directed causal curve"

if each $\dot{\gamma}(t) \in T_{\gamma(t)}(M)$ is a future-directed non-spacelike vector.

Note: These are all paths that matter or massless particles (light, gluons etc) can travel.

Note: Here, $\dot{\gamma}(t) = 0$ is allowed because non-spacelikeness merely requires $g(\dot{\gamma}, \dot{\gamma}) \leq 0$.

Definition:

Past-directed timelike or causal curves are defined analogously.

The past & future of an event

Definition:

Mnemonic help: $I^+(p)$ is the set of events an actual chronometer could reach from p.

The "chronological future" of a $p \in M$ is the set

$$I^+(p)$$

of events that can be reached from p on a future-directed timelike curve.

Note: $I^+(p)$ is always an open subset of M (because if γ is a timelike future-directed curve from p then any sufficiently small perturbation of γ is also a timelike future-directed curve starting from p.)

We will assume that there are no closed timelike curves. For travellers on such curves, life would repeat itself but we know that everybody ages, so it can't. I.e. on such spacetimes there'd be no thermodynamic arrow of time in whose direction entropy increases.

Note: Since $\gamma = 0$ is excluded we have

$$p \notin I^+(p)$$

$\longrightarrow$ except if there is a closed timelike curve through p.

Definition:

In $J^+(p)$ is any event that could be causally affected by what happened at p.

The "causal future" of a $p \in M$ is the set

$$J^+(p)$$

of events that can be reached from p on a future-directed causal curve.

Note: Clearly, we always have $p \in J^+(p)$ because $\mu = 0$ is allowed.

Definitions:

- Analogously, one defines the chronological past $I^-(p)$ and the causal past $J^-(p)$.

- One defines the pasts and futures of a set S of events through:

$$I^{\pm}(S) := \bigcup_{p \in S} I^{\pm}(p), \quad J^{\pm}(S) := \bigcup_{p \in S} J^{\pm}(p)$$

- Their boundaries are denoted: $\dot{I}^{\pm}(S)$, $\dot{J}^{\pm}(S)$

Example: Assume $S := \{$events travelled by immortal observer$\}$
Then $\dot{J}^-(S) =$ event horizon of this observer.

Relation to geodesics:

- The above definitions do not refer to geodesics.

- For Minkowski space, clear:

  $I^+(p)$ = set of events reachable by future-directed timelike geodesics from p.

  $\dot{I}^+(p)$ = set of events reachable by future-directed null geodesics from p.

  But this is not true in general spacetimes!

  Intuition: E.g. singularities can be in the way of a geodesic.

- But locally, the situation is as for Minkowski space:

Theorem:

Assume $p \in M$ and $U \subset M$ is a convex normal neighborhood of p.

(Any 2 pts in U are connected by a unique geodesic.)

Then: a) $I^+(p)\big|_U$ = set of events reachable by future-directed timelike geodesics from p.

(means "restricted to")

b) $\dot{I}^+(p)\big|_U$ = set of events reachable by future-directed null geodesics from p.

c) If $q \in J^+(p) - I^+(p)$ then any causal curve between p and q is a null geodesic.

Recall:

- The "chronological future" of a set S is the set $I^+(S)$ of events that can be reached from S on a future-directed timelike curve.

- The "causal future" of a set S is the set $J^+(S)$ of events that can be reached from S on a future-directed causal curve.

  recall: includes null curves.

- In Minkowski space: $J^+(S) = I^+(S) \cup \dot{I}^+(S)$

Recall: $\dot{I}^+(S)$ is the boundary of $I^+(S)$

Properties of $\dot{I}^+(S)$, in general?

Definition:

A subset $Q \subset M$ is called "achronal" if no two points in Q can be connected by a future-directed time-like curve, i.e., by a curve that, e.g., a clock (with mass) could travel. Thus, $Q \in M$ is achronal iff:

$$I^+(Q) \cap Q = \phi$$

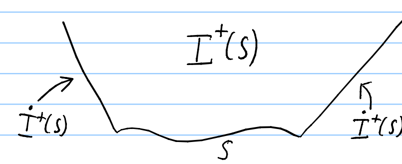
empty set

Theorem:

For all $S \subset M$, the set

$$\dot{I}^+(S)$$

(if not empty) is an achronal 3-dimensional submanifold of M.



Example: In Minkowski space, if S is a point p, then $\dot{I}^+(p)$ is the boundary of the light cone.

Indeed, no two points of the boundary of the lightcone are connected by time-like paths.

- In general spacetimes, however:

  It is clear that

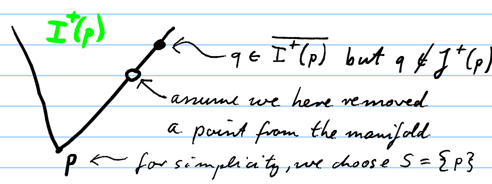
$$\overline{J^+(S)} = \overline{I^+(S)}$$

← bar denotes closure of the set

  but we notice that, generally:

$$J^+(S) \neq \overline{I^+(S)}$$

- Example:



$\Rightarrow$ here, $q \in \overline{I^+(p)}$, but $q \notin J^+(p)$ because there is no nonspacelike curve between p and q.

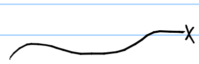
⇝ Idea:

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as indicator for the absence or existence of a singularity.

Definition:

- We say that a point $p \in M$ is future (past) endpoint of a curve γ if

  $\forall$ neighborhoods $\mathcal{U}$ of p there exists a $t_0 \in \mathbb{R}$ so that



$$\gamma(t) \subset \mathcal{U} \quad \forall\, t > t_0 \qquad (t < t_0 \text{ for past endpoint})$$

- Note: p need not be reached by γ!

and every point $p \in \mathcal{M}$ is an element of open sets $\subset \mathcal{M}$.

- Since $p \in \mathcal{M}$, it is always possible to extend a curve γ continuously beyond p. Note: $p \in \mathcal{M}$!
  $\Rightarrow$ The existence of an endpoint here indicates not a singularity, or 'hole' in the manifold, but merely that we chose to end the curve before it is necessary!

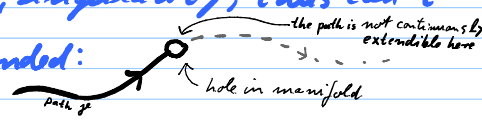
Definition:

- We say that a curve γ is future (past) inextendible if it does not possess a future (past) endpoint.
- Intuition: If γ future inextendible, then:

  a.) γ runs to ∞, or

  b.) γ runs around forever, or

  c.) γ hits a hole, i.e., singularity, thus can't be continuously extended:



Theorem:

Assume $\mathcal{C} \subset \mathcal{M}$ is closed and assume:

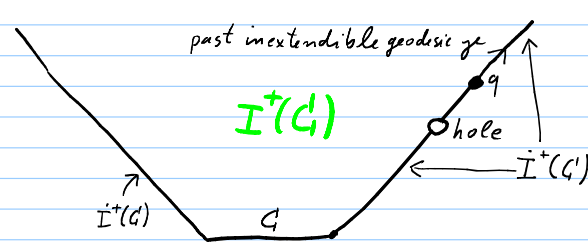
$$q \in \dot{I}^+(\mathcal{C}) \quad , \quad q \notin \mathcal{C}$$

Then, q lies on a null geodesic γ inside $\dot{I}^+(\mathcal{C})$ and the curve γ either:

a.) has past endpoint on $\mathcal{C}$

or b.) is past inextendible (because meets hole)

Example for b:

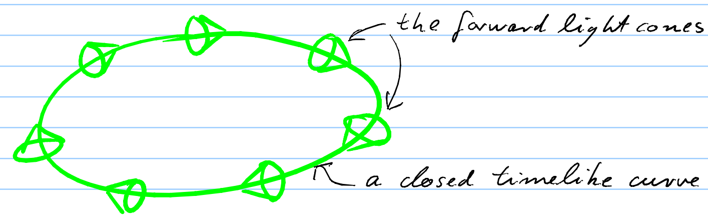


Strategy:

- Study inextendible curves!
- This includes these cases:
  - a.) γ hits singularity – will be main interest!
  - b.) γ running off to ∞.
  - c.) γ going round and round forever.

⇝ Must address potential causality problems of case c)

Example:



Note: This is not a problem with time-orientability here!