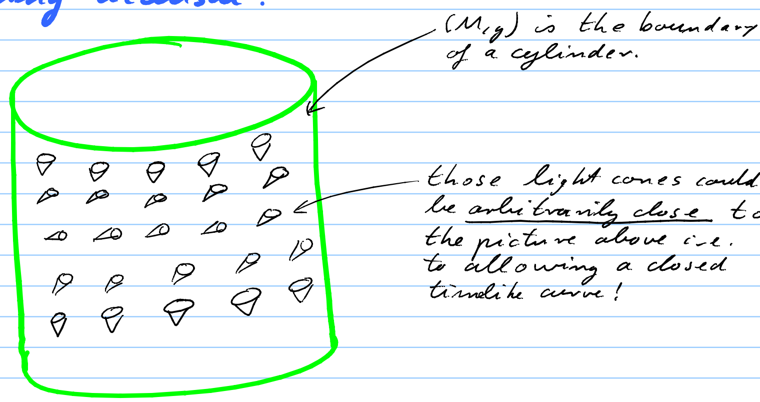
Causality

- We say that (M,g) is "causal" if it does not contain closed causal (i.e. time or null) curves.

  Problem: (M,g) may nevertheless be arbitrarily close to being acausal:



→ We say that a spacetime (M,g) is "strongly causal", if

  $\forall p$ and $\forall$ neighborhoods $\mathcal{U}$ of p there is a neighborhood $V \subset \mathcal{U}$ so that:

  No causal curve γ intersects V more than once.

- **Indeed:**

  If (M,g) is not strongly causal $\Rightarrow$ there exists a causal curve γ which comes arbitrarily close to intersecting itself.

- → We require strong causality to keep causal curves at least a finite distance from intersecting themselves.

Problem:

Still, arbitrarily small perturbations in the metric, somewhere, could allow causal curves to self-intersect!

Solution:

a) Consider perturbing the metric g through

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} - w_\mu w_\nu$$

with a time-like cotangent vector field.

↑ needed for theorem 2 below.

b) Notice: $\tilde{g}_{\mu\nu}$ still has same signature but light cones are now "wider":

Compare $v^\mu v^\nu \tilde{g}_{\mu\nu}$ and $v^\mu v^\nu g_{\mu\nu}$:

$$v^\mu v^\nu \tilde{g}_{\mu\nu} = v^\mu v^\nu g_{\mu\nu} \underbrace{- v^\mu w_\mu v^\nu w_\nu}_{\text{always } < 0} < v^\mu v^\nu g_{\mu\nu}$$

Thus, it is easier for vectors v to be timelike or null for $\tilde{g}$ than for g.

$\Longrightarrow$

$(M, \tilde{g})$ has all the causal curves of (M, g), and more!

c) Define:

(M, g) is called "stably causal", if there exists a w so that even $(M, \tilde{g})$ is causal.

Theorem 1: (M,g) stably causal $\Rightarrow$ (M,g) strongly causal.

Theorem 2: (M,g) stably causal

$\Updownarrow$

There exists a differentiable function $f \in \mathcal{F}(M)$ so that ∇f is a past-directed time-like vector field.

Remark: This means that f can be viewed as a cosmic "clock". (It is not unique, however)

Recall: Time-orientability $\Leftrightarrow$ $\exists$ past-pointing smooth timelike vector field. (which need not be a gradient field)

The plan: We assume that spacetime is stably causal. (so travellers cannot go on cyclic paths)

Therefore, inextendible paths (i.e. paths without endpoint $p \in M$) either:

a.) go to ∞, or

b.) end in a singularity

$\rightsquigarrow$ Continue to study inextendible curves

$\rightsquigarrow$ Arrive at key concepts of Cauchy horizon and global hyperbolicity.

Recall:

- We considered the set of points $J^+(S)$ ("the causal future") that can somehow be reached from a set S. (i.e. the set of points that are affected by S)
- Now consider set of points that can only be reached from S: (i.e. the set of events that depend on S and only S)

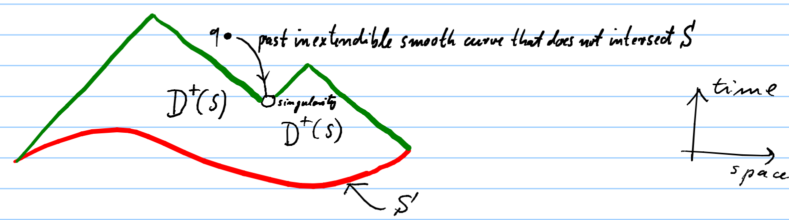
Definition:

i.e., a set of events among which no object could travel

Assume $S \subset M$ is a closed achronal set.
Then, the "future domain of dependence of S"
is defined as:

$$D^+(S) := \left\{ p \in M \;\middle|\; \begin{array}{l} \text{Every past inextendible causal} \\ \text{curve through } p \text{ intersects } S \end{array} \right\}$$

Example:



Why $q \notin D^+(S)$? Some of its past inextendible causal curves do not intersect S because they get stuck at the hole!

(q is affected by events in the "shadow" of the singularity)

Definition:

Analogously, the "past domain of dependence of S" is:

$$D^-(S) := \left\{ p \in M \;\middle|\; \begin{array}{l} \text{Every future inextendible causal} \\ \text{curve through } p \text{ intersects } S \end{array} \right\}$$

(the set of events p that affect only S)

Definition:

The "full domain of dependence of S" is:

$$D(S) := D^+(S) \cup D^-(S)$$

Definition:

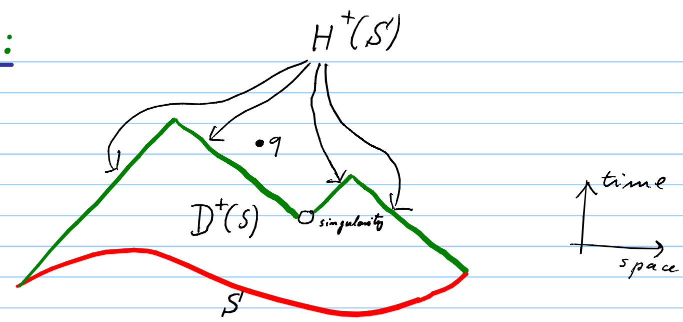
(set of latest events that are affected only by S? How far have initial conditions on S full predictive power?

The "future Cauchy horizon of S", denoted $H^+(S)$ is:

$$H^+(S) := \overline{D^+(S)} - \underbrace{I^-}_{\text{chronological past}}(D^+(S))$$

(Note: $\Rightarrow H^+(S)$ is achronal. Why?)

Example:



analogously:

Definition:

The "past Cauchy horizon of S'", denoted $H^-(S)$

is: $H^-(S') := \overline{D^-(S)} - I^+(D^-(S))$ (set of earliest events that affect only S)

Definition:

The "full Cauchy horizon of S'" is defined as:

$$H(S') := H^+(S) \cup H^-(S)$$

Proposition:

$$H(S') = \dot{D}(S')$$

Definition:

A closed, achronal set S' is called a "Cauchy surface", if its full Cauchy horizon vanishes, i.e., if

a) $H(S) = \emptyset$ ← empty set or equivalently if

b) $\dot{D}(S) = \emptyset$ or equivalently if

c) $D(S') = \mathcal{M}$

Note: This follows Wald. The definitions by others (Hawking, Ellis, Geroch et al.) are equivalent, but more tedious.

Remarks:

- Cauchy surfaces are important because if the conditions on a Cauchy surface are known, then everything on M can be predicted and retrodicted.
  Note: E.g., anti de Sitter space has no Cauchy surfaces!

- Since a Cauchy surface is achronal, it can be viewed as an "instant in time".

- The term "surface" is motivated by a **theorem:**

  Every Cauchy surface, Σ, is a 3-dimensional C^0 submanifold of M.

Definition:

If (M,g) possesses a Cauchy surface then it is called "*globally hyperbolic*".

Remark: We'll need this notion later for a cosmological singularity theorem.

Proposition:

If (M,g) is globally hyperbolic, then:

- There exists a "global time function f" so that every surface of constant f is a Cauchy surface.

- (M,g) is stably (and therefore also strongly) causal.

Recall: Plan is to study inextendible geodesics in order to detect singularities.

Now: How to identify these geodesics which are inextendible because they end at a singularity in the manifold?

First: Avoid trivial cases where manifold is ending but could be extended.

Definition:

We say that $(\mathcal{M}, g)$ is inextendible, if it is not isometric to a proper subset of another spacetime $(\mathcal{M}', g')$.

$\rightsquigarrow$ We will always assume that $(\mathcal{M}, g)$ is inextendible.

Definition:

A geodesic which is inextendible but possesses a finite range of its affine parameter is called "incomplete".

Note: This is to exclude inextendible geodesics which keep going to ∞.

Definition:

We say that $(\mathcal{M}, g)$ possesses a "**singularity**" if it possesses an incomplete geodesic.

$\rightsquigarrow$ We distinguish singularities of null, spacelike and timelike type.

- When going along an incomplete geodesic towards a "singularity", 3 things can happen:

  I) A scalar constructed from $R^{\mu\nu}{}_{\rho\sigma}$, e.g. R, $R^{\mu\nu}R_{\mu\nu}$, etc diverges.

  ⇝ We say it is a "scalar curvature singularity".

  II) In a parallel transported tetrad frame, a scalar component of $R^{\mu\nu}_{\rho\sigma}$ or its covariant derivatives diverge.

  ⇝ We say it is a "parallel-propagated curvature singularity"

  III) None of the above. Example: "Conical singularity". (cut out a suitable piece and identify the boundaries of the cut)

  (This way mfld can be diffable while some paths cannot.)

  ⇝ We say it is a non-curvature singularity

Fundamental problem:

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are obviously present.
- But these spacetimes are highly symmetric.

  Do more realistic, i.e. perturbed spacetimes also show these singularities?

Example:

Spherically symmetric dust shell infall.

In Newton gravity: Use catastrophe theory

$\Rightarrow$ e.g., predict ∞ mass density to occur, but not if symmetry perturbed!

In Einstein gravity: Use singularity theorems

$\Rightarrow$ e.g., predict black hole singularity to occur, even if symmetry is perturbed, (if assuming e.g. dominant energy cond. etc.)

or also: post-dict a cosmological singularity

Remark: Black holes provide finite energy endpoint of grav. collaps, thus stabilizing GR energetically. *Note:* In QM, charge driven collaps is bounded at finite energy by uncertainty principle.

Remark:

Singularity theorems $\Rightarrow$ prediction of singularities is robust.

Thus: If quantum gravity is to resolve singularities, it will have to overcome this robustness!

Strategy for singularity theorems:

a) Focus attention on singularities that can be identified by the existence of incomplete inextendible timelike (or null) geodesics.

Why? It is clear that these are important singularities because observers travelling such a geodesic have their eigentime bounded above and/or below.

Other singularities? (e.g. singularities identified through incomplete spacelike geodesics or singularities identified by some other criterion.)

May well exist in addition but the standard singularity theorems do not attempt to predict them too.

b.) Basic idea:

Singularities can be in the way of geodesics.

$\Longrightarrow$

The presence of singularities interferes with the property of geodesics of being extremal length curves.

c.) Recall:

Extremizing curve length $\xrightarrow{\text{(Euler Lagrange equation)}}$ geodesic equation

The geodesic equation is a differential equation.

Thus:

At least locally, geodesics are paths of extremal length:

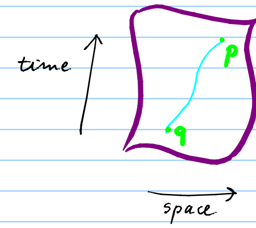
- Space-like geodesics are curves of shortest proper distance.
- Time-like geodesics are curves of maximal proper time (i.e. of maximal eigentime).

Why maximal?

If there is a timelike curve between two events p, q, then there are timelike curves with shorter eigentime: just take a longer (spatially) path and travel it faster.

d.) Prove that, even in generic spacetimes:

There always exist curves of maximal length between two events.



What assumptions are needed?

E.g., the assumption that spacetime is globally hyperbolic suffices.

e.) Further assume that matter obeys a suitable energy condition, (usually the so-called strong energy condition) and use it to prove that geodesics meet a divergence of a quantity called expansion, Θ, in finite proper time.

$\Rightarrow$ these extremal length curves cannot be geodesics with eigentime larger than a certain finite amount either into the past or future.

f.) Conclude that there are incomplete geodesics, i.e., that we have a singularity in the past (or future).

A singularity theorem:

Assume that:

- (M, g) is a globally hyperbolic spacetime

- The energy-momentum tensor of matter obeys the "Strong energy condition":

$$\left(T_{\mu\nu} - \frac{1}{2} T^{\rho}{}_{\rho}\, g_{\mu\nu}\right) \xi^{\mu} \xi^{\nu} \geq 0 \quad \forall \text{ timelike } \xi.$$

Notice: Since the Einstein equation can be brought in the form $kR_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}$, the strong energy condition is a condition on the Ricci tensor too. This will be the use of the strong energy condition.

- There exists a C^2 spacelike Cauchy surface Σ, on which the trace of the extrinsic curvature, K, is bounded from above by a negative constant C:

$$K(p) \leq C < 0 \quad \text{for all } p \in \Sigma$$

Then:

No past-directed timelike curve from a spacelike hypersurface Σ can have eigentime, i.e., proper length, larger than $\frac{3}{C}$.

I.e.: All past-directed timelike geodesics are incomplete.

$\Longrightarrow$ There is a cosmological singularity in the finite past!

because all past-directed paths end on it.