A singularity theorem:

Assume that:

- (M, g) is a globally hyperbolic spacetime
- The energy-momentum tensor of matter obeys the **Strong energy condition**:

$$\left(T_{\mu\nu} - \frac{1}{2} T^{\varsigma}{}_{\varsigma}\, g_{\mu\nu}\right) \xi^{\mu} \xi^{\nu} \geq 0 \quad \forall \text{ timelike } \xi.$$

Notice: Since the Einstein equation can be brought in the form $kR_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}$, the strong energy condition is a condition on the Ricci tensor too. This will be the use of the strong energy condition.

- There exists a C^2 spacelike Cauchy surface Σ, on which the trace of the extrinsic curvature, K, is bounded from above by a negative constant C:

$$K(p) \leq C < 0 \quad \text{for all } p \in \Sigma$$

Then:

No past-directed timelike curve from a spacelike hypersurface Σ can have eigentime, i.e., proper length, larger than $\frac{3}{C}$.

I.e.: All past-directed timelike geodesics are incomplete.

because all past-directed paths end on it.

$\Rightarrow$ There is a cosmological singularity in the finite past!

Extrinsic curvature?

later more on this

- The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between the spacelike hypersurface and the time dimension.

  Intuitively: it is the rate of the expansion of spacetime, more precisely its negative, the rate of contraction.

  Thus: Assuming $K(p) \leq C < 0$ ($\forall p \in \Sigma$) meant that spacetime has a finite minimum expansion rate everywhere on Σ.

  ⇝ We'll define expansion below in detail.

The strong energy condition?

Recall:

- The "weak energy condition":

  $$T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v: \ g(v,v) < 0$$

  Meaning? For an observer with unit tangent v the local energy density is: $T_{\mu\nu} v^\mu v^\nu \geq 0$

- The "dominant energy condition":

  $$\underbrace{T_{\mu\nu} v^\mu v^\nu \geq 0}_{\text{weak energy condition}} \quad \text{and} \quad K_\mu K^\mu \leq 0$$

  ↳ i.e. $T_{\mu\nu} v^\nu$ is non-space-like.

  where v is any timelike vector and $K_\mu := T_{\mu\nu} v^\nu$

  Meaning? The local energy-momentum flow vector K may not be conserved but has to be non-space-like: Flow should be into the future ← need for causality.

The "strong energy condition"

Matter is said to obey the strong energy condition iff:

$$\left(T_{\mu\nu} - \frac{1}{2} T^{\rho}{}_{\rho}\, g_{\mu\nu}\right) \xi^{\mu} \xi^{\nu} \geq 0 \quad \forall \text{ timelike } \xi.$$

- **Intuition?** Excludes matter that causes accelerated expansion. (as we will discuss below)
- **Plausible?** Yes, obeyed by known matter. (but not by dark energy)
- **Relationship?** Independent of weak and dominant energy condition.

Concretely: For known matter, $T_{\mu\nu}$ is diagonalizable to obtain:

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & p_1 & & \\ & & p_2 & \\ 0 & & & p_3 \end{pmatrix}$$

ρ: energy density observed by comoving observer

p_1, p_2, p_3: principal pressures

The energy conditions then read:

- **Weak:** $\rho \geq 0$ and $\rho + p_i \geq 0$ for $i \in \{1,2,3\}$
- **Dominant:** $\rho \geq |p_i|$ for $i \in \{1,2,3\}$
- Exercise: Show this $\longrightarrow$ **Strong:** $\rho + \sum_{i=1}^{3} p_i \geq 0$ and $\rho + p_i \geq 0$ for $i \in \{1,2,3\}$

  Note: could possibly be also negative.

Recall: A cosmological constant Λ can be viewed as a contribution to $T_{\mu\nu}$.

Indeed, there is no big bang singularity, e.g., if $w = -1 \ \forall t$, i.e., in de Sitter spacetime inflation $a(t) = e^{Ht}$.

Exercise: Show that the strong energy condition is violated in cosmology iff $w < -\frac{1}{3}$, i.e., iff the expansion is accelerating: $\ddot{a}(t) > 0$.

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, θ, in finite proper time:

important notion also e.g. in study of grav. collaps of stars.

The "expansion", θ:

- Consider a "congruence of timelike geodesics" (e.g., freely falling dust.) through Σ, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper time, their tangent vector field ξ, namely

$$\xi := \frac{d}{d\tau}$$ ← proper time

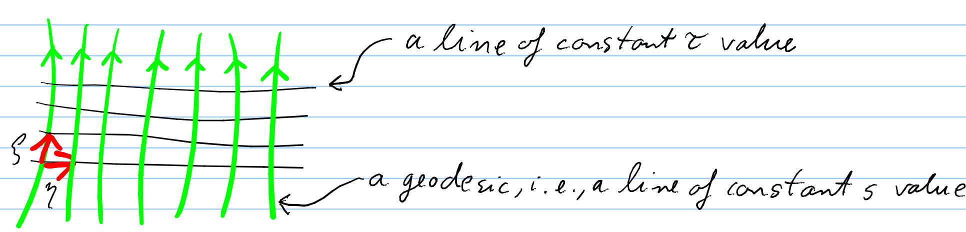
will obey: $g(\xi,\xi) = -1 \ \forall p$.

- Consider now a one-parameter subfamily of these geodesics:

$$\gamma(\tau, s)$$ ← parameter of family of neighboring geodesics.

Then, we define the deviation vector (a "connecting vector field"):

$$\eta := \frac{d}{ds}$$



□ How does η change along a geodesic?

τ, s are Riemann normal coordinates for a geodesic traveller.

$$\Rightarrow \frac{d}{d\tau}\frac{d}{ds} = \frac{d}{ds}\frac{d}{d\tau}, \text{ i.e., } [\xi, \eta] = 0$$

□ Since the torsion vanishes: $0 = T(\xi,\eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi,\eta]$

$$\Rightarrow \quad \nabla_\xi \eta = \nabla_\eta \xi$$

$$\Rightarrow \quad \xi^\mu \nabla_{e_\mu} \eta^\nu e_\nu = \eta^\alpha \nabla_{e_\alpha} \xi^\beta e_\beta$$

$$\Rightarrow \quad \xi^\mu \eta^\nu{}_{;\mu} e_\nu = \eta^\alpha \xi^\beta{}_{;\alpha} e_\beta$$

$$\Rightarrow \quad \xi^\mu \eta^\nu{}_{;\mu} = \eta^\alpha \xi^\nu{}_{;\alpha} = \eta^\alpha B^\nu{}_\alpha \text{ for } \boxed{B^\nu{}_\mu := \xi^\nu{}_{;\mu}}$$

$\Rightarrow$ Along the geodesic's direction, ξ, the deviation vector η^μ changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

□ The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely into:

$$B_{\mu\nu} = \underset{\text{antisymmetric}}{\omega_{\mu\nu}} + \overset{\text{symmetric and trace}=0}{\sigma_{\mu\nu}} + \underset{\text{rest}}{t_{\mu\nu}}$$

Cosmic ballet tensor field.

(all 3 terms are tensors because the split is covariant)

We have: $\omega_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$, clearly.

But $\sigma_{\mu\nu}, t_{\mu\nu} = ?$

In preparation: define the projector $h_{\mu\nu}$ onto $(\mathbb{R}\xi)^\perp$ i.e. onto the spatial components:

is timelike

$$h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$$

Check: is $h_{\mu\nu} w^\nu$ really always $\perp$ to ξ?

Indeed: $\xi^\mu h_{\mu\nu} w^\nu = (\xi, w) + \overbrace{(\xi,\xi)}^{=-1}(\xi,w) = 0$

Define: The "expansion", θ, is defined as the magnitude of the spatial part of B:

$$\theta := B^{\mu\nu} h_{\mu\nu}$$

Claim: $Tr(B) = \theta$

Indeed:
$$\theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \xi^{\mu}\xi_{\nu} B_{\mu}{}^{\nu}$$
$$= Tr(B) + \xi^{\mu} \underbrace{\xi_{\nu} \nabla_{\mu} \xi^{\nu}}$$

($= 0$ because $\nabla_{\xi}\xi = 0$ for geodesics.)

Therefore:

$$\sigma_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3}\theta h_{\mu\nu}$$

(because: $Tr(h_{\mu\nu}) = g^{\mu\nu} h_{\mu\nu} = g^{\mu\nu}(g_{\mu\nu} + \xi_{\mu}\xi_{\nu}) = 4 - 1$)

the part of $B_{\mu\nu}$ which is symmetric and traceless.

and:

$$t_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu}$$ ← the "rest term".

Interpretation:

a) $\omega_{\mu\nu}$ is antisymmetric: $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$\Rightarrow$ it generates Lorentz transformation for η.

but all η are $\perp$ to the time direction

$\Rightarrow$ $\omega_{\mu\nu}$ generates spatial rotations of neighboring geodesics around another. So, $\omega_{\mu\nu}$ is called

$\omega =$ "Twists tensor"

One can prove: (nontrivial)

If one chooses the congruence of geodesics $\perp$ to Σ then $\omega_{\mu\nu} = 0$.

b.) $\sigma_{\mu\nu}$ is symmetric, $\sigma_{\mu\nu} = \sigma_{\nu\mu}$. (i.e. hermitean)

Consider "diagonalized", by suitable choice of cd basis.

$\Rightarrow$ $\sigma_{\mu\nu}$ changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow $\rightarrow$ become points on an ellipsoid.

infinitesimal transport along geodesics

Note: Since $Tr(\sigma) = 0$ we have $det(e^{\lambda \sigma}) = 1$

finite transport

$\Rightarrow$ The volume spanned by basis vectors stays the same under the action of σ.

$\rightsquigarrow$ **Definition:** $\sigma_{\mu\nu} =:$ "Shear tensor" 

c.) While the twist and shear tensors are both traceless and therefore volume-preserving, we see that the trace part, θ, i.e., more precisely

$$t_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} =: \text{"Expansion tensor"}$$

recall: is projector on spatial part.

is indeed generating the spatial expansion or contraction of nearby geodesics!

Evolution of θ along a geodesic?

Recall:

Given, in particular, the strong energy condition, our singularity theorem claimed that geodesics meet a divergence of a quantity called expansion, θ, in finite proper time in the past and this will mean a big bang singularity:

important notion also e.g. in study of grav. collaps of stars.

The "expansion", θ:

- Consider a "congruence of timelike geodesics" through Σ, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$: (Σ is a Cauchy surface)

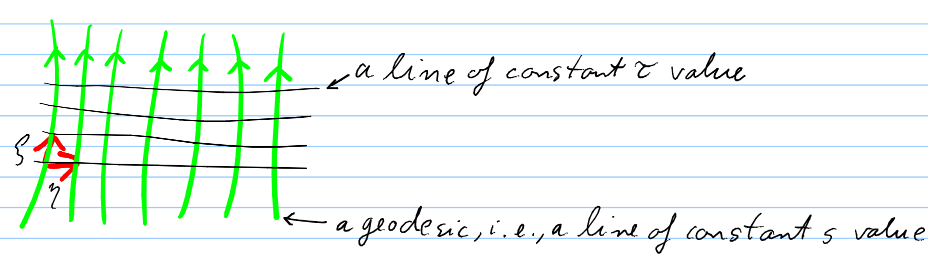
  e.g., freely falling dust.

- We consider a one-parameter sub-family of these geodesics:

$$\gamma(\tau, s)$$

τ: eigentime

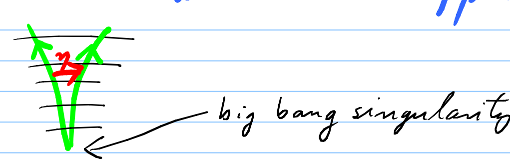
s: parameter of family of neighboring geodesics.



- Then, we define the deviation vector to a neighboring geodesic:

$$\eta := \frac{d}{ds}$$

- The singularity theorem claims that this happened in the past:



How does η change along a past-directed timelike geodesic with tangent ξ?

We showed:

$$\boxed{\xi^\mu \eta^\nu{}_{;\mu} = \eta^\alpha B^\nu{}_\alpha \quad \text{where} \quad B^\nu{}_\mu := \xi^\nu{}_{;\mu}}$$

$\Rightarrow$ Along the geodesic, ξ, the deviation vector η^μ changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

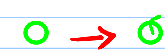
▫ The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely:

$$B_{\mu\nu} = \underbrace{\omega_{\mu\nu}}_{\text{antisymmetric}} + \overbrace{\sigma_{\mu\nu}}^{\text{symmetric and trace}=0} + \underbrace{t_{\mu\nu}}_{\text{rest}}$$

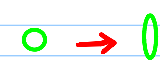
Explicitly:

Volume preserving →

$$\omega_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} - B_{\nu\mu}\right) \qquad \text{Twist:}$$

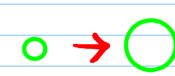


$$\sigma_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} + B_{\nu\mu}\right) - \frac{1}{3}\theta h_{\mu\nu} \qquad \text{Shear:}$$



Volume changing:

$$t_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} \qquad \text{Expansion:}$$



Here, we defined: $\theta := B^{\mu\nu} g_{\mu\nu}$ and $h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$

I.e., the Expansion, θ, is the trace of B, which we showed is also equal to the magnitude of the spatial part of B: $\theta = B^{\mu\nu} h_{\mu\nu}$.

Key question: $\boxed{\text{What is the dynamics of } \theta \text{ ?}}$

The Raychaudhuri equation

For the derivation, we will use:

A) Definition of B is: $B_{\mu\nu} := \xi_{\mu;\nu}$

B) The curvature tensor obeys the Ricci equation:

$$\boxed{\xi^a{}_{;bc} - \xi^a{}_{;cb} = R^a{}_{bcd}\,\xi^d}$$

C) ξ is tangent to a geodesic, i.e., it obeys: $\nabla_\xi \xi = 0$

i.e.: $0 = \nabla_{\xi^a e_a} \xi^b e_b = \xi^a \nabla_{e_a} \xi^b e_b = \xi^a \xi^b{}_{;a} e_b$

True for all e_a, thus: $\boxed{\xi^a \xi^b{}_{;a} = 0}$

Now calculate the rate of change of B along the geodesic:

$$\underbrace{\xi^c B_{ab;c}}_{\nabla_\xi B} \overset{(A)}{=} \xi^c \xi_{a;bc}$$

$$\overset{(B)}{=} \xi^c \xi_{a;cb} + \xi^c R_{abcd}\,\xi^d$$

$$\overset{\text{Leibnitz rule}}{=} \underbrace{(\xi^c \xi_{a;c})_{;b}}_{=0} - \xi^c{}_{;b}\,\xi_{a;c} + R_{abcd}\,\xi^c \xi^d$$

$$\overset{(C)}{=} -\xi^c{}_{;b}\,\xi_{a;c} + R_{abcd}\,\xi^c \xi^d$$

$$\overset{(A)}{=} -B^c{}_b\,B_{ac} + R_{abcd}\,\xi^c \xi^d$$

In summary, we derived:

$$\xi^c B_{ab;c} = -B^c{}_b B_{ac} + R_{abcd}\,\xi^c \xi^d \qquad (*)$$

The trace of (*) will be the Raychandhuri equation.

But first, we recall:

$$\square\ \xi = \frac{d}{d\tau}$$

$$\square\ \mathrm{Tr}\,B = B_{\mu\nu}\, g^{\mu\nu} = \theta$$

$\Rightarrow$ Trace(LHS) of (*) reads $\frac{d}{d\tau}\theta$!

Now on the RHS of (*) use the decomposition

$B_{\mu\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu}$ to express $B^c{}_b B_{ac}$:

$$B^c{}_b B_{ac} = \omega^c{}_b\left(\underline{\omega_{ac}} + \sigma_{ac} + \tfrac{1}{3}\theta h_{ac}\right)$$

$$+ \sigma^c{}_b\left(\omega_{ac} + \underline{\sigma_{ac}} + \tfrac{1}{3}\theta h_{ac}\right)$$

$$+ \tfrac{1}{3}\theta h^a{}_b\left(\omega_{ac} + \sigma_{ac} + \underline{\tfrac{1}{3}\theta h_{ac}}\right)$$

When taking the trace, $g^{ab}B^c{}_b B_{ac}$, only the diagonal terms survive:

Exercise: show it is 3

$$\mathrm{Tr}(BB) = g^{ab}B^c{}_b B_{ac} = \omega_{ab}\omega^{ab} + \sigma_{ab}\sigma^{ab} + \frac{1}{9}\theta^2 \overbrace{h_{ab}h^{ab}}^{3 \,=}$$

The Raychandhuri equation is then the trace of Eq. (*) :

recall: Ricci tensor is $R_{cd} = R_{cda}{}^{a}$

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \underbrace{\sigma_{ab}\sigma^{ab}}_{\text{always positive}} - \underbrace{\omega_{ab}\omega^{ab}}_{\substack{\text{always positive} \\ \text{(and vanishes if} \\ \text{choose congruence } \perp \Sigma)}} - \underbrace{R_{cd}\,\xi^c \xi^d}_{\text{pos. or neg?}}$$

?

Dynamics

- Assume that

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \quad \text{for all timelike } \xi$$

i.e., using the Einstein equation

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^a{}_a\right)$$

we are assuming that

$$T_{\mu\nu}\xi^\mu\xi^\nu - \tfrac{1}{2}\underbrace{\xi^\mu\xi_\mu}_{=-1} T \geq 0 \quad \text{whenever } \xi^\mu\xi_\mu < 0$$

i.e. the **Strong Energy Condition**.

Thus, assuming the strong energy condition:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0$$

i.e., $$-\frac{1}{\theta^2}\frac{d\theta}{d\tau} - \frac{1}{3} \geq 0$$

i.e., $$\boxed{\frac{d}{d\tau}\theta^{-1} \geq \frac{1}{3}} \qquad (+)$$

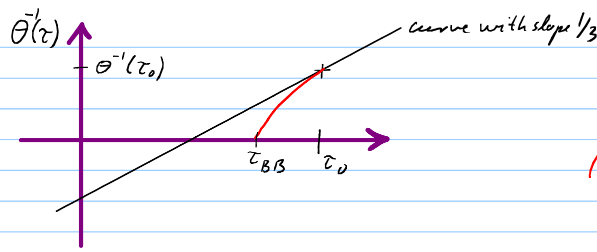
Consider the cases when the geodesics are initially all either

a.) diverging, i.e., $\theta(\tau_0) > 0$ (expanding universe) or

b.) converging, i.e., $\theta(\tau_0) < 0$ (contracting universe)

(This is reformulating the theorem's assumption that the extrinsic curvature (i.e. the expansion or contraction at some time exceeds a certain finite value everywhere)

a.)

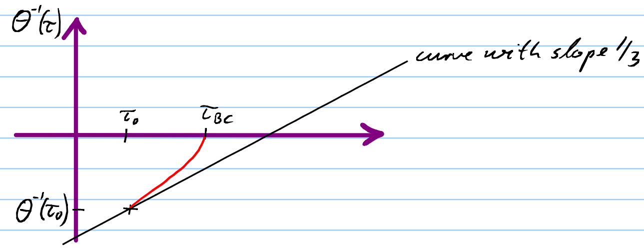


τ_0 = e.g. today

(red curve) = curve $\theta^{-1}(\tau)$ of slope $\geq \frac{1}{3}$

We see that $\theta^{-1}(\tau)$ must have hit $\theta^{-1}(\tau)=0$ at a finite time τ_{BB} (Big Bang).

b)



τ_0 = e.g. today

(red curve) = curve of slope $\geq 1/3$

We see that $\theta^{-1}(\tau)$ will hit $\theta^{-1}(\tau)=0$ at a finite time τ_{BC} (Big Crunch)

Conclusion:

Eq. (+) implies that $\theta^{-1}(\tau)$ must go through 0, i.e.:

a.) for sufficiently early τ, have $\theta \to +\infty$, i.e.: Big Bang

b.) for sufficiently late τ, have $\theta \to -\infty$, i.e.: Big Crunch

Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.