GR for Cosmology, Achim Kempf — **Lecture 24**

Note Title

Problem: In general relativity, what are the effective generalizations of the conservation laws for energy, momentum and angular momentum, e.g., to calculate collisions of galaxies?

Source: NASA

Why important? E.g., to probe for dark matter!

$\approx 23\%$. Compare: visible matter: $\approx 5\%$, dark energy: $\approx 72\%$

Recall: The tetrad formalism's advantages are, e.g.:

- Allows one to choose bases in tangent spaces independently from any choice of coordinates
- Can have $g_{\mu\nu}(v) = \eta_{\mu\nu}$, which allows one to use the usual γ matrices $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and obtain the Dirac equation.
- Re-express GR in terms of tetrads e^a_μ as a gauge theory
  $\rightsquigarrow$ Starting point for quantum gravity, e.g. Loop Quantum Gravity

Also: Tetrad formalism of tensor-valued forms lends itself to issues that require integration, such as the question whether there are still effective global conservation laws.

Global conservation laws:

Recall:

- In special relativity, energy and momentum conservation etc express the fact that the translation in time or space (or rotation etc) of a solution is a solution too.
  This doesn't hold in curved space-time, of course.

- It is always true that

$$T^{\mu\nu}{}_{;\nu} = 0$$

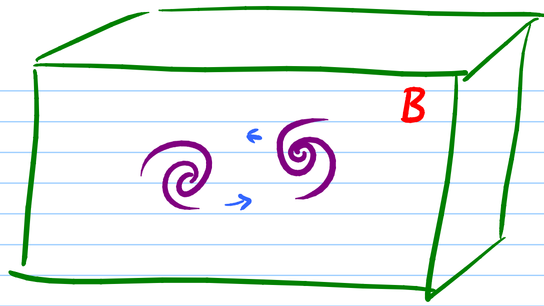
  but this is not a proper conservation law.

- Only if a space-time does not change in a particular direction, i.e., if it possesses a Killing vector field, at least locally, then solutions go into solutions and one does obtain a conservation law.

Idea:

- In regions of significant gravitational effects, e.g., where two galaxies interact with another there is surely no Killing vector field.

- But, if we consider the system in a box,



Note: The box B is a 3-dim spatial region (spatial hypersurface)

which is big enough so that space-time is essentially flat where the box boundaries are, then:

- We expect that the box has a "total box energy" a "total box momentum" and "total box angular momentum" which are conserved in time.
- Why? From Newton we know e.g. that total kinetic plus gravitational potential energy are conserved.

Problems:

a.) What is "gravitational potential energy in GR"?

Recall: Locally, gravity can always be eliminated, $\Gamma^{\mu}{}_{\nu\rho}(p)=0$, i.e. there surely is no local notion of gravitational potential energy!

b.) We must expect that, in GR, "gravitational potential energy", if it exists in some sense, can also enter or escape the box in the form of gravitational waves.

a) "Gravitational potential energy":

- There is no such thing, locally, e.g., as a tensor.
- But, we can pursue this **Strategy**:

I) Reformulate the Einstein equation

$$-\frac{1}{2}\underbrace{H_{\alpha\beta\gamma}}_{*(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma)} \wedge \Omega^{\beta\gamma} = 8\pi G *T_\alpha$$

so that it reads:

These so-called Landau-Lifshitz differential 3-forms play the rôle of gravitational potential energy-momentum.

(that's where (tensor-valued) differential forms come in handy)

$$d(\text{something}) = 8\pi G \sqrt{g}\,(*T_\alpha + \overbrace{*t_\alpha})$$

(We will have to show that the Einstein equation can be written this way!)

II) Defining $\tau_\alpha := T_\alpha + t_\alpha$ we then have:

$$d(\text{something}) = 8\pi G \sqrt{g} * \tau_\alpha$$

III) Then, from $d^2 = 0$ we obtain:

$$d\left(\sqrt{g} * \tau_\alpha\right) = 0$$

IV) Using Stokes' theorem, $\int_D dv = \int_{\partial D} v$, we obtain:

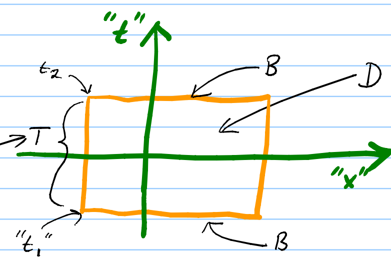
$$0 = \int_D \overbrace{d\left(\sqrt{g} * \underbrace{\underbrace{\tau_\alpha}_{1\text{-form}}}_{3\text{-form}}\right)}^{4\text{ form}} = \int_{\partial D} \overbrace{\sqrt{g} * \tau_\alpha}^{3\text{-form}} \qquad \otimes$$

D: 4dim. space-time region

∂D: 3-dim space time sub manifold.

V) Choose D bounded by the large box B, i.e., on large scales we have:

(T is assumed far out in space, where there is no matter, energy, momentum and no curvature.)



so that equation ⊗, namely

$$\int_{\partial D} v_\xi * \tau_\alpha = 0$$

becomes:

$$0 = \int_{B(t_1)} v_\xi * \tau_\alpha + \int_{B(t_2)} v_\xi * \tau_\alpha + \int_{T} v_\xi * \tau_\alpha$$

VI) We notice that

$$\int_{T} v_\xi * \tau_\alpha = 0$$

if, as we here assume, space is flat and empty far out in space. I.e., where events of T are, there we have $\underset{\uparrow}{T_\alpha = 0}$ and $\underset{\uparrow}{t_\alpha = 0}$

b/c no matter b/c no curvature

VII) Therefore:

$$\int_{B(t_1)} v_\xi * \tau_\alpha + \int_{B(t_2)} v_\xi * \tau_\alpha = 0$$

But we notice that the 2nd integral has the timelike normal vector to $B(t_2)$ pointing to the past.

$\Rightarrow$ If we define the integrations both with respect to future-pointing normals, we obtain:

$$\int_{B(t_1)}^{f} \sqrt{g}\, *\tau_\alpha = \int_{B(t_2)}^{f} \sqrt{g}\, *\tau_\alpha$$

VIII) Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$P_\mu := \int_B \sqrt{g}\, *\tau_\mu$$

B ← big box

It is conserved in time:

$$\boxed{P_\mu(t_1) = P_\mu(t_2)}$$

(Because under
$\theta(x)^\mu \to A^\mu{}_\nu(x)\, \theta^\nu(x)$
we have generally
$\omega(x) \to A(x)\omega(x)A^{-1}(x) - (dA)A$
but far out in space we now have:
$A(x) \to$ const. i.e.
$\omega(x) \to A(x)\omega(x)A^{-1}(x) - 0$)

Note: It is a Minkowski tensor with respect to local Lorentz transformations that approach a constant Lorentz transformation far out in space.

Determination of τ_α

- Recall starting assumption, namely that we can reformulate the Einstein equation

$$-\frac{1}{2} H_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = 8\pi G\, *T_\alpha$$

so that it reads:

$$d(\text{something}) = 8\pi G \sqrt{g}\, \big(\overbrace{*T_\alpha + *t_\alpha}^{*\tau_\alpha \,=}\big)$$

There are choices to be made here.

Then $d^2 = 0$ yields: $d\left(\sqrt{g}\,(*T_\alpha + *t_\alpha)\right) = 0$

$\Rightarrow$ Conservation law via Gauß' theorem.

Q: The choice of $d(\text{something})$ and, correspondingly of t_α is not unique.

How to fix this choice?

A: In order to be able to define also an angular momentum, we'll need $T_{\mu\nu} = T_{\nu\mu}$.

Proposition: There is a unique decomposition so that

$$t_\alpha = t_{\alpha\beta}\,\theta^\beta$$

is symmetric. For this decomposition:

"Landau-Lifshitz 3-forms":

$$\underbrace{*\overset{\text{1-form}}{t^\alpha}}_{\text{3-form}} = -\frac{1}{16\pi G}\,\underbrace{H^{\alpha\beta\gamma\delta}}_{\text{0-form},}\underbrace{\left(\omega_{\sigma\beta}\wedge\omega^\sigma{}_\gamma\wedge\theta_\delta - \omega_{\beta\gamma}\wedge\omega_{\sigma\delta}\wedge\theta^\sigma\right)}_{\text{3-form}}$$

$$H^{\alpha\beta\gamma\delta} := *(\theta^\alpha\wedge\theta^\beta\wedge\theta^\gamma\wedge\theta^\delta)$$

Sketch of proof:

Einstein equation: $-\frac{1}{2}\Omega_{\beta\gamma}\wedge H^{\beta\gamma}{}_\alpha = 8\pi G *T_\alpha$

2^{nd} structure equation: $\Omega_{\beta\gamma} = d\omega_{\beta\gamma} - \omega_{\sigma\beta}\wedge\omega^\sigma{}_\gamma$

$$\Rightarrow \underbrace{-\frac{1}{2}d\omega_{\beta\gamma}\wedge H^{\beta\gamma}}_{=\,-\frac{1}{2}d(\omega_{\beta\gamma}\wedge H^{\beta\gamma}) - \frac{1}{2}\omega_{\beta\gamma}\wedge dH^{\beta\gamma}{}_\alpha} + \frac{1}{2}\omega_{\sigma\beta}\wedge\omega^\sigma{}_\gamma = 8\pi G *T_\alpha \qquad (*)$$

Re-write, using (recall):

$$0 = DH^{\beta\gamma}{}_\alpha = dH^{\beta\gamma}{}_\alpha + \omega^\beta{}_\sigma\wedge H^{\sigma\gamma}{}_\alpha + \omega^\gamma{}_\sigma\wedge H^{\beta\sigma}{}_\alpha - \omega^\sigma{}_\alpha\wedge H^{\beta\gamma}{}_\sigma$$

$\Rightarrow$ Einstein equation:

$$-\frac{1}{2}d(\omega_{\beta\gamma}\wedge H^{\beta\gamma}) + \frac{1}{2}\omega_{\beta\gamma}\wedge\left(\omega^\beta{}_\sigma\wedge H^{\sigma\gamma}{}_\alpha + \omega^\gamma{}_\sigma\wedge H^{\beta\sigma}{}_\alpha - \omega^\sigma{}_\alpha\wedge H^{\beta\gamma}{}_\sigma\right) + \frac{1}{2}\omega_{\sigma\beta}\wedge\omega^\sigma{}_\gamma = 8\pi G *T_\alpha$$

Notice: It is of the form $d(\text{something}) = 8\pi G(*T_\alpha + *t_\alpha)$

Final step (see Straumann): Add terms to make $T_{\alpha\beta}$ symmetric.

$\Rightarrow$ We now have all ingredients to calculate the conserved ADM energy-momentum vector

$$P_\mu := \int_{B \leftarrow \text{big box}} \sqrt{g}\, * \tau_\mu$$

(The "positive energy theorem") with $\tau_\mu = T_\mu + t_\mu$

t_μ from gravity using Eqn. (G) above.

T_μ from matter

Theorem: If the dominant energy condition holds, then $P_0 \geq 0$

(i.e. the ADM 4-vector is future-directed timelike or lightlike: $P_\mu P^\mu \leq 0$ and $p_0 \geq 0$)

Angular momentum?

- Choose coordinates that become cartesian Minkowski far out.
- **Define:** $*M^{\alpha\beta} := x^\alpha * \tau^\beta - x^\beta * \tau^\alpha$
- **Proposition:** $d\left(\sqrt{g}\, * M^{\alpha\beta}\right) = 0$

Note: For this it is necessary to have chosen the definition of t which has $t_{\alpha\beta}$, and therefore also $\tau_{\alpha\beta}$, symmetric.

$\Rightarrow$ ADM 4-angular momentum $J^{\alpha\beta} := \int_B \sqrt{g}\, * M^{\alpha\beta}$ is conserved!

b.) Taking account of grav. waves:

- Do we have to account for possible energy-momentum loss due to radiation from the region of strong gravity, in particular, grav. wave radiation?
- This depends on how we define our "box".

  If the box is large enough for our assumptions to hold, then grav. radiation cannot escape the region between t_1, t_2
- **But also:** We can choose space-like hypersurfaces which at large distances "bend up" to become asymptotically light-like.
- This leads to the Sachs Bondi 4 momentum $P_\mu^{(SB)}$.