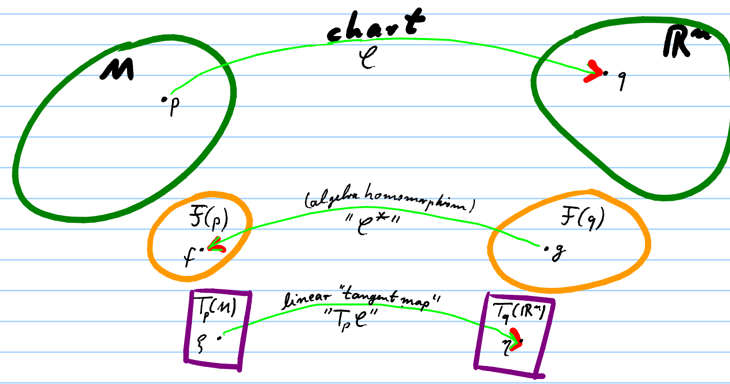
Note Title

GR for Cosmology, Achim Kempf, Lecture 3

The "physicist's definition of $T_p(\mathcal{M})$"

Recall: We obtain concrete representations for $p \in \mathcal{M}$ and $f \in \mathcal{F}(p)$ and $\xi \in T_p(\mathcal{M})$ using a chart $\varphi: \mathcal{M} \to \mathbb{R}^n$:



Recall: Def's used pre-composition:

$$\varphi^*[g] = g \circ \varphi$$

$$T_p\varphi[\xi] = \xi \circ \varphi^*$$

Terminology: φ^* is called the "pullback" of φ

$T_p\varphi$ is called the "pullback" of φ^*

Namely:

- Each $p \in \mathcal{M}$ has now a concrete image $q \in \mathbb{R}^n$, i.e., it has 'coordinates'.
- Each $f \in \mathcal{F}(p)$ is the image of a concrete function germ $g \in \mathcal{F}(q)$.
- Each $\xi \in T_p(\mathcal{M})$ has now a concrete image

$$\eta \in T_q(\mathbb{R}^n)$$

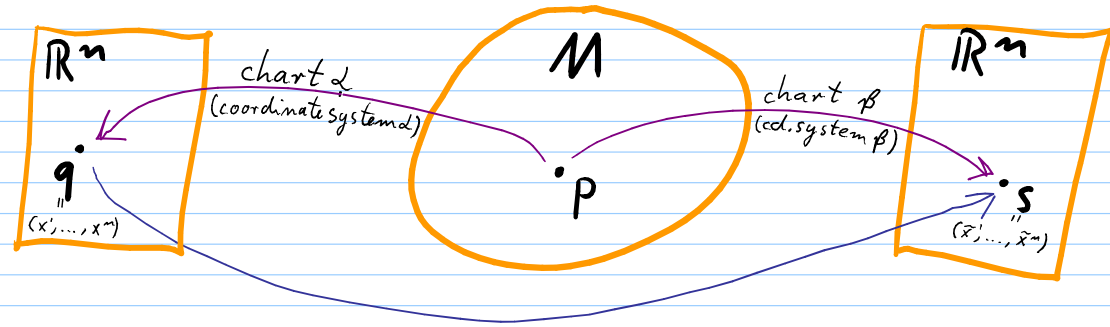
which we know has the form:

coefficients $\in \mathbb{R}$

$$\eta = \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial x^i}\bigg|_{x=q}$$

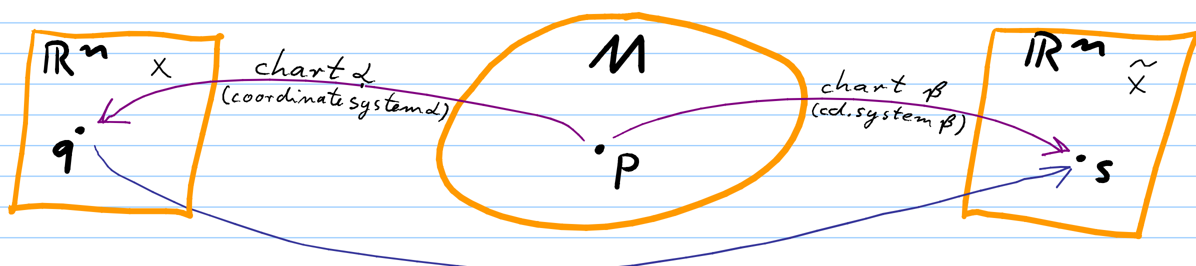
Question:

Given a $p \in M$ and a $\xi \in T_p(M)$, how do their coordinates and coefficients change under a change of charts?

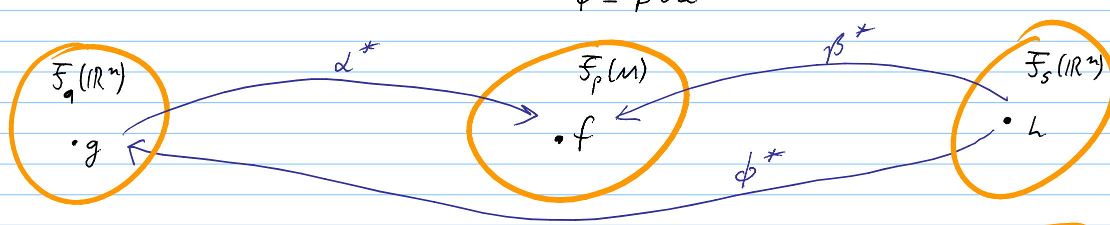


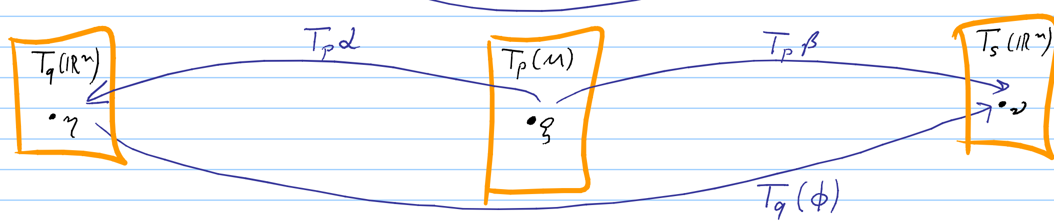
$\phi = \beta \circ \alpha^{-1}$, i.e.: $\phi : \mathbb{R}^n \to \mathbb{R}^n$

$\Rightarrow$ When changing from chart α to chart β:



$\phi = \beta \circ \alpha^{-1}$





1. Every point $p \in M$ now has 2 images,
$q = (x^1, \dots, x^n)$ and $s = (\tilde{x}^1, \dots, \tilde{x}^n)$

$$(\tilde{x}^1, \dots, \tilde{x}^n) = \phi(x^1, \dots, x^n)$$

Concretely: $\tilde{x}^i = \phi^i(x^1, \dots, x^n)$.

2. Every function germ $f \in \mathcal{F}_p(M)$ has 2 pre-images,
$g \in \mathcal{F}_q(\mathbb{R}^n)$ and $h \in \mathcal{F}_s(\mathbb{R}^n)$, related by
$f(p) = g(q) = h(s)$ $\quad (\in \mathbb{R})$ and by

$$h(\tilde{x}^1, \dots, \tilde{x}^n) = g(x^1, \dots, x^n) \qquad (*) \quad \text{(in a neighborhood)}$$

3. Every tangent vector $\xi \in T_p(M)$ now has 2 images,
$\eta \in T_q(\mathbb{R}^n)$ and $v \in T_s(\mathbb{R}^n)$.

By construction: (b/c of precomposition)

$$\eta(g) = \xi(f) = v(h) \qquad (\in \mathbb{R})$$

$\Rightarrow$ in particular:

$$\overbrace{\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} g(x^1, \dots, x^n)}^{\eta(g)}\Big|_{x=q} = \overbrace{\sum_{j=1}^{n} v^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1, \dots, \tilde{x}^n)}_{\substack{\shortparallel \\ g(x^1, \dots, x^n)}}}^{v(h)}\Big|_{\tilde{x}=s} \quad \text{by } (*)$$

$$= \sum_{\substack{j=1 \\ k=1}}^{n} v^j \frac{\partial x^k}{\partial \tilde{x}^j}\Big|_{\tilde{x}=s} \frac{\partial}{\partial x^k} g(x^1, \dots, x^n)\Big|_{x=q}$$

Must be true for all g!

$$\Rightarrow \qquad \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} = \sum_{\substack{j=1 \\ i=1}}^{n} \nu^j \left.\frac{\partial x^i}{\partial \tilde{x}^j}\right|_{\tilde{x}=s} \frac{\partial}{\partial x^i}$$

The $\left\{\frac{\partial}{\partial x^i}\right\}$ are linearly independent.

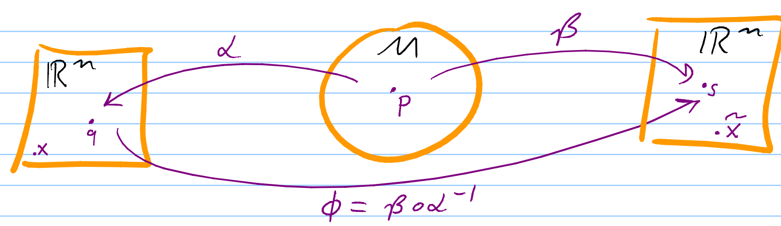
Jacobian matrix $D\phi^{-1}$ of ϕ^{-1} at s.

$$\Rightarrow \qquad \eta^i = \sum_{j=1}^{n} \left.\frac{\partial x^i}{\partial \tilde{x}^j}\right|_{\tilde{x}=s} \nu^j$$

Jacobian matrix $D\phi$ of ϕ at q.

$\Rightarrow$ conversely:
$$\nu^i = \sum_{j=1}^{n} \left.\frac{\partial \tilde{x}^i}{\partial x^j}\right|_{x=q} \eta^j$$

Summary:



Given $\xi \in T_p(M)$, its images in charts α, β, namely $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}$ and $\nu = \sum_{i=1}^{n} \nu^i \frac{\partial}{\partial \tilde{x}^i}$, are related by

Jacobian matrix $D\phi$

$$\underbrace{\nu^i = \sum_{j=1}^{n} \left.\frac{\partial \tilde{x}^i}{\partial x^j}\right|_{x=q} \eta^j}_{} = \sum_{j=1}^{n} \left.\frac{\partial \phi^i(x^1, \dots, x^n)}{\partial x^j}\right|_{x=q} \eta^j$$

This transformation property can also be used as the starting point for a definition of tangent vectors!

with: $\tilde{x}^i = \phi^i(x^1, \dots, x^n)$

The "physicist's definition of $T_p(M)$"

Def: A tangent vector $\xi^{(phys)} \in T_p^{(phys)}(M)$ is a map that assigns to each (germ of a) chart a coefficient vector $\in \mathbb{R}^n$, so that if

- $(\eta^1, \dots, \eta^n)$ is coefficient vector w. resp. to chart α
- $(v^1, \dots, v^n)$ is coefficient vector w. resp. to chart β

then:

$$v^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}\bigg|_{x=\beta(p)} \eta^j \qquad \text{with } \tilde{x}^i = \phi^i(x), \quad \phi = \beta \circ \alpha^{-1}$$

So far, 2 equiv. defs. of $T_p(M)$:

In a chart, α, a tangent vector, $\xi \in T_p(M)$ is:

- algebraically: $\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=\alpha(p)}$

  i.e. it is a directional derivative

  Defining property: Leibniz rule.

- physically: $(\eta, \dots, \eta^n)$

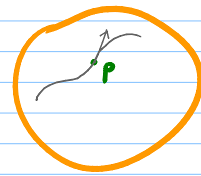
  i.e. it is just the direction vector,

  Defining property: chart change transformation rule

Finally:

The "geometric definition of $T_p(M)$":

Idea: Tangent vectors as tangents to paths.



Consider paths in M that pass through p:

$$\gamma : \mathbb{R} \to M$$

$$\gamma(0) = p$$

Note: For any $f : M \to \mathbb{R}$, we obtain:

$$f \circ \gamma : \mathbb{R} \to \mathbb{R}$$

Define:

Two diffable paths, γ_a, γ_b are called equivalent, if for all $f \in \mathcal{F}_p(M)$:

$$\frac{d}{dt}\left(f \circ \gamma_a\right)\Big|_{t=0} = \frac{d}{dt}\left(f \circ \gamma_b\right)\Big|_{t=0} \qquad \otimes$$

Intuition: Two paths γ_a, γ_b are equivalent if they have the same 'velocity' at p:

Note: this includes speed and direction because $\otimes$ must hold for all $f \in \mathcal{F}_p(M)$.

Definition: $T_p(M)^{(geom)}$ is the set of equivalence classes of diffable paths through p.

we'll usually mean $T_p^{(alg)}(M)$ when we write $T_p(M)$.

Are $T_p(M)^{(geom)}$ and $T_p(M)^{(alg)}$ equivalent?

Yes!

really: each equivalence class of diff'able paths through p

- Each path γ defines a linear map $\bar{\gamma}$:

$$\bar{\gamma}: \mathcal{F}(p) \longrightarrow \mathbb{R}$$

$$\bar{\gamma}: f \to \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

- These $\bar{\gamma}$ obey the Leibniz rule:

$$\bar{\gamma}(fg) = \frac{d}{dt}(f \cdot g)(\gamma(t))\Big|_{t=0} = \frac{d}{dt}\big(f(\gamma(t))\, g(\gamma(t))\big)\Big|_{t=0}$$

$$= \frac{d}{dt} f(\gamma(t))\Big|_{t=0} g(\underbrace{\gamma(0)}_{=p}) + f(\gamma(0)) \frac{d}{dt} g(\underbrace{\gamma(t)}_{=p})\Big|_{t=0}$$

$$= \bar{\gamma}(f)\, g + f\, \bar{\gamma}(g) \checkmark$$

- $\Rightarrow$ $\bar{\gamma}$ is an element of $T_p^{(alg)}(M)$

The "Cotangent Space" $T_p(M)^*$:

Recall:

Given an n-dimensional vector space V, the set of linear maps $\omega: V \to \mathbb{R}$ forms also an n-dim. vector space. It is called the "dual space", and denoted V^*.

Definition:

The dual vector space to $T_p(M)$ is called the Cotangent Space, and denoted $T_p(M)^*$.

We notice: For every (germ of a) function at p,

$$f \in \mathcal{F}(p)$$

one naturally obtains an element

$$\text{"}df\text{"} \in T_p(M)^*$$

called the "differential of f."

Namely:

$df: T_p(M) \to \mathbb{R}$ is the linear map:

$$df: \xi \to \xi(f)$$

(Note: thus, we can view "d" as a map: $d: \mathcal{F}_p(M) \to T_p(M)^*$. See later...)

Concretely: in a cds., i.e., in a chart,

the abstract $\xi \in T_p(M)$ and $f \in \mathcal{F}(p)$

correspond to some $\eta \in T_q(\mathbb{R}^n)$ and $g \in \mathcal{F}(q)$.

Then: $\underset{\underset{T_p(M)^*}{\cup}}{dg}: T_q(\mathbb{R}^n) \to \mathbb{R}$

$$dg: \quad \eta \to \eta(g) = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\bigg|_{x=q} g(x^1, \ldots, x^n)$$

Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\bigg|_{x=q}$

a basis of $T_q(\mathbb{R}^n)$ is $\left\{ \frac{\partial}{\partial x^i}\bigg|_{x=q} \right\}_{i=1}^{n}$

Question: What is the dual basis in $T_q(\mathbb{R}^n)^*$?

- Consider the coordinate functions: $x^k : \mathbb{R}^n \to \mathbb{R}$.
- Their differentials $dx^k \in T_q(\mathbb{R}^n)^*$ obey:

$$dx^k : T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$dx^k : \left.\frac{\partial}{\partial x^i}\right|_{x=q} \to \left.\frac{\partial}{\partial x^i} x^k\right|_{x=q} = \delta_i^k$$

$\Rightarrow$ The dual basis in $T_q(\mathbb{R}^n)^*$ is given by

$$\{dx^k\}_{k=1}^n$$

Thus:

Every element $\omega \in T_q(\mathbb{R}^n)^*$ takes the form:

$$\omega = \sum_{i=1}^n \underbrace{\omega_i}_{\in \mathbb{R}} dx^i$$

and its action is:

$$\omega : T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$\omega : \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} \to \sum_{i=1}^n \omega_i dx^i\left(\sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j}\right)$$

$$= \underbrace{\sum_{i=1}^n \omega_i \sum_{j=1}^n \eta^j \underbrace{\frac{\partial}{\partial x^j} x^i}_{=\delta_j^i}}_{=\sum_{i=1}^n \omega_i \eta^i}$$

$$\Rightarrow \omega\left(\sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n \omega_i \eta^i \qquad \text{(I)}$$

In particular: For arbitrary $g \in \mathcal{F}(q)$, its differential $dg \in T_q(\mathbb{R}^m)^*$ must be of the form:

$$dg = \sum_{k=1}^{m} \omega_k \, dx^k \text{ with suitable } \omega_k \in \mathbb{R}.$$

How to calculate them?

We know:

$$dg(\eta) = \eta(g) = \sum_{i=1}^{m} \eta^i \underbrace{\frac{\partial}{\partial x^i} g(x)\Big|_{x=q}}_{=\omega^i} \quad (\mathrm{II})$$

Compare I, II $\Rightarrow$ $\omega_i = \frac{\partial}{\partial x^i} g(x)\Big|_{x=q}$

$$\Rightarrow \boxed{dg = \sum_{i=1}^{m} \left(\frac{\partial}{\partial x^i} g(x)\Big|_{x=q} \right) dx^i}$$

Exercise: (the "pull back" map)

Assume that $\mathfrak{g} \in T_p(M)^*$, under two charts α, β, as above, corresponds to $\omega \in T_q(\mathbb{R}^m)^*$ and $\mu \in T_s(\mathbb{R}^m)^*$ with:

$$\omega = \sum_{i=1}^{m} \omega_i \, dx^i \quad \text{and} \quad \mu = \sum_{i=1}^{m} \mu_i \, d\tilde{x}^i$$

Show that $$\mu_i = \sum_{j=1}^{m} \frac{\partial x^j}{\partial \tilde{x}^i}\Big|_{\tilde{x}=s} \omega_j$$

Notice that this is the inverse of the Jacobian matrix of $\beta \circ \alpha^{-1}$ at q

Remark: The physicist's definition of $T_p(M)^*$ uses this.

Some notation and terminology:

- Elements of $T_p(M)$ are called contravariant vectors
- Elements of $T_p(M)^*$ are called covariant vectors
- One often writes symbolically

$$\xi = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x^i}\Big|_p \quad \text{for } \xi \in T_p(M)$$

$$\omega = \sum_{i=1}^{m} \omega_i \, dx^i \quad \text{for } \omega \in T_p(M)^*$$

even without specifying a particular chart.

We are now ready to define tensors:

Def: A tensor, t, of rank (r,s) is an element of

$$T_p(M)^r_s := \underbrace{T_p(M)\otimes \ldots \otimes T_p(M)}_{r \text{ factors}} \otimes \underbrace{T_p(M)^* \otimes \ldots \otimes T_p(M)^*}_{s \text{ factors}}$$

In a chart:

$$t = \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_s}=1}^{m} \underset{\substack{\cap \\ \mathbb{R}}}{t^{i_1\ldots i_r}_{j_1\ldots j_s}} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}$$

Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$\tilde{t}^{i_1\ldots i_r}_{j_1\ldots j_s} = \sum_{\substack{l_1\ldots l_s \\ k_1\ldots k_r}=1}^{m} \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial \tilde{x}^{j_s}} \, t^{k_1\ldots k_r}_{l_1\ldots l_s}$$

Thus: $T_p(M) = T_p(M)^1$ and $T_p(M)^* = T_p(M)_1$, i.e.:

- a tangent vector is a tensor of rank $(1,0)$
- a cotangent vector is a tensor of rank $(0,1)$

Finally: From local to global!

Def: We call $T(M) := \bigcup_{p \in M} (p, T_p(M))$, the **Tangent bundle**.

a "base point" (p); a "fibre" ($T_p(M)$)

Note: $T(M)$ is itself a manifold. It is $2n$-dimensional.

Def: $T(M)$ is then also called the "Total Space".

Def: M is also called the "Base Space".

Recall that all $T_p(M)$ are n-dimensional real vector spaces, i.e., are isomorphic to $\mathbb{R}^n$.

Def: We therefore call $\mathbb{R}^n$ the "Standard Fibre".

Remark: One obtains other Fibre bundles by choosing other standard fibers.

E.g.:
- Co-tangent bundle $T^*(M)$
- (r,s)-tensor bundle $T^r_s(M)$
- Bundles for isospinors (vector bundles) and gauge groups (principle bundles)

Def: The map $\pi : T(M) \rightarrow M$

$$\pi : (p, T_p(M)) \rightarrow p \qquad \left(\text{i.e.: } \pi^{-1}(p) = T_p(M)\right)$$

is called the "Bundle Projection".

Def: A Section, σ, is a map, $\sigma : M \rightarrow T(M)$, which is a continuous right inverse of π:

$$\pi(\sigma(x)) = x \quad \forall x \in M \qquad (\text{i.e.: } \pi \circ \sigma = id)$$

Notice: The graph of a "field" is a section of its fibre bundle.

Recall: The graph of a function $f: A \to B$ is:

$$\{(a, f(a))\}_{a \in A}$$

Def:

- A tangent vector field is a map $\xi : \overset{\in M}{p} \to \overset{\in T_p(M)}{\xi_p}$

  In a chart: $\xi = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i}$

- A cotangent vector field is a map $\omega : \overset{\in M}{p} \to \overset{\in T_p^*(M)}{\omega_p}$

  In a chart: $\omega = \sum_{i=1}^{n} \omega_i(x)\, dx^i$

- Similarly, tensor fields: $t : \overset{\in M}{p} \to \overset{\in T_p(M)^r_s}{t_p}$

  In a chart: $t = \sum t^{j_1 \dots j_r}_{i_1 \dots i_s}(x) \frac{\partial}{\partial x^{j_1}} \cdots \frac{\partial}{\partial x^{j_r}} dx^{i_1} \cdots dx^{i_s}$

So far, not a concern with GR, but it does come up with gauge theories.

Why then fibre bundles? To capture global nontriviality.

Fibre bundles are required to be locally trivial:

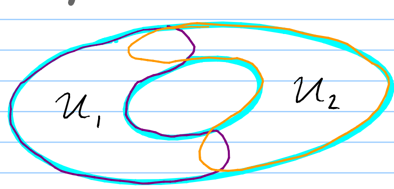
M can be covered with neighborhoods U_r, so that

means: there exists a differentiable isomorphism

or other standard fibre for other fibre bundles.

$$\pi^{-1}(U_r) \simeq U_r \times \mathbb{R}^n$$

But fibre bundles are allowed to be globally nontrivial:



For a suitable vector bundle B, we can have

$$\pi^{-1}(U_1) \simeq U_1 \times \mathbb{R}^n$$

$$\pi^{-1}(U_2) \simeq U_2 \times \mathbb{R}^n$$

but in the overlap regions, the two isomorphisms may differ $\Rightarrow B \not\simeq M \times \mathbb{R}^n$

(The isomorphisms may differ by elements of $Gl_n(\mathbb{R})$, the "structure group" here)

Definition: For the algebra of C^∞ functions $\mathcal{M} \to \mathbb{R}$ we write $\mathcal{F}(\mathcal{M})$.

Note: One can show that contravariant vector fields are the derivations of the algebra $\mathcal{F}(\mathcal{M})$, i.e.:

If ξ is a contravariant vector field, then

$$\xi : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$$

is linear and obeys the Leibniz rule:

$$\xi(fg) = \xi(f)\,g + f\,\xi(g)$$

for all $f, g \in \mathcal{F}(\mathcal{M})$.

Next topic: Differential forms:

We already have covered some differential forms:

- The set $\Lambda_0 := \mathcal{F}(\mathcal{M})$ is called the set of 0-forms.
- The set of covariant vector fields is denoted Λ_1 and called the set of 1-forms.
- For $r = 2, 3, \dots$ the set, Λ_r, of r-forms is defined to be the set of totally anti-symmetric tensor fields of rank $(0, r)$.