Recall:

- The set $\Lambda(M)$ of differential forms on M is an associative algebra, called the Grassmann algebra over M.

- The multiplication in $\Lambda(M)$ is the wedge product: $\wedge: \Lambda_s(M) \times \Lambda_r(M) \rightarrow \Lambda_{s+r}(M)$

- The exterior derivative $d: \Lambda(M) \rightarrow \Lambda(M)$ is an anti-derivation of degree $K=1$ of the Grassmann algebra $\Lambda(M)$.

But: How to obtain a directional derivative on $\Lambda(M)$?

Recall: Tangent vectors ξ are directional derivatives on $\Lambda_0(M)$!

Plan now:

A. Define an anti-derivation i_ξ of degree $K=-1$: the inner derivation.

(i_ξ will generalize feeding a tangent vector ξ to a 1-form to feeding it to a p-form.)

B. Combine d, i_ξ to obtain a derivation of degree $K=0$: the Lie derivative

(And the Lie derivative is going to be the directional derivative for differential forms and tensors)

A. The "Inner Derivation":

- Assume ξ is a tangent vector field.
- Our aim: to define an anti-derivation, i_ξ, of degree $K=-1$, i.e., a linear map

$$i_\xi : \Lambda_s(M) \to \Lambda_{s-1}(M)$$
$$i_\xi : \quad \omega \to i_\xi(\omega)$$

which obeys the anti-Leibniz rule:

$$i_\xi(\omega \wedge \nu) = i_\xi(\omega)\wedge \nu + (-1)^r \omega \wedge i_\xi(\nu)$$

if $\omega \in \Lambda_r(M)$.

- Definition:

$$i_\xi : \Lambda_0 \to 0$$
$$i_\xi : \Lambda_1 \to \Lambda_0$$
$$i_\xi : \omega \to \omega(\xi)$$

- Recall: By linearity and the anti-Leibniz rule this already defines $i_\xi : \Lambda(M) \to \Lambda(M)$.

- Proposition: If $\chi \in \Lambda_s(M)$ then $i_\xi(\chi) \in \Lambda_{s-1}(M)$ maps $(s-1)$ tangent vectors $\eta_1, \dots, \eta_{s-1}$ this way:

$$i_\xi(\chi)(\eta_1, \eta_2, \dots, \eta_{s-1}) := \chi(\xi, \eta_1, \eta_2, \dots, \eta_{s-1})$$

Example: * Consider $\underset{\Lambda_2(M)}{\chi} := \underset{\Lambda_1(M)}{\omega} \wedge \underset{\Lambda_1(M)}{\nu}$

* What is $i_\xi(\chi) \in \Lambda_1(M)$? Leibniz rule $\Rightarrow$

$$i_\xi(\chi) = i_\xi(\omega \wedge \nu) = i_\xi(\omega) \wedge \nu + (-1)^1 \omega \wedge i_\xi(\nu)$$
$$= \omega(\xi)\nu - \nu(\xi)\omega$$

* Apply $i_\xi(\chi) \in \Lambda_1(M)$ to a tangent vector η:

$$i_\xi(\chi)(\eta) = \omega(\xi)\nu(\eta) - \nu(\xi)\omega(\eta)$$

* Compare with claim of proposition:

$$i_\xi(\chi)(\eta) = i_\xi(\omega \wedge \nu)(\eta) = i_\xi(\omega \otimes \nu - \nu \otimes \omega)(\eta)$$
$$= \omega(\xi)\nu(\eta) - \nu(\xi)\omega(\eta) \quad ✓$$

Recall: $\omega \wedge \nu = \omega \otimes \nu - \nu \otimes \omega$

Properties of i_ξ:

- $i_{\xi_1} \circ i_{\xi_2} = - i_{\xi_2} \circ i_{\xi_1}$

(Exercise: prove this)

- Thus, in particular:

$$i_\xi \circ i_\xi = 0$$

- **Recall:** We also have $d \circ d = 0$

Recall: For $\xi \in T_p(M)$, $\chi \in T_p^*(M)$, we have $i_\xi(\chi) = \chi(\xi) = \xi(\chi)$

(Simply the evaluation of a dual vector applied to a vector in the vector space)

Definition: The inner derivation, $i_\xi(\chi)$, of a $\chi \in \Lambda(M)$ is also called the interior product of ξ and χ.

B. The Lie derivative "L_ξ": (algebraic definition)

Vectors $\xi : \Lambda_0(M) \to \Lambda_0(M)$ are directional derivatives.

How to generalize the notion of directional derivative to all of $\Lambda(M)$?

We have:
- $d : \Lambda_s(M) \to \Lambda_{s+1}(M)$ generalizes the notion of differential $d: \Lambda_0 \to \Lambda_1$, $d: f \to df$ to all of $\Lambda(M)$.
- $i_\xi : \Lambda_s(M) \to \Lambda_{s-1}(M)$ generalizes the notion of evaluation of vectors ξ on covectors $\omega \in \Lambda_1(M)$ to all of $\Lambda(M)$.

$$\boxed{\text{Spoiler: It will be: } L_\xi = d \circ i_\xi + i_\xi \circ d}$$

To construct L_ξ, let us first collect desired properties:

- As a directional derivative, it should be a derivation, not an anti-derivation, i.e.:

$$L_\xi(\omega \wedge \nu) = L_\xi(\omega) \wedge \nu + \omega \wedge L_\xi(\nu)$$

(Recall that the directional derivatives on functions $\Lambda_0(M)$, namely the tangent vectors, are mapping $\Lambda_0(M) \to \Lambda_0(M)$)

- L_ξ should map r-forms into r-forms:

$$L_\xi : \Lambda_r(M) \to \Lambda_r(M)$$

i.e. it should be of degree $K = 0$. In particular:

- On functions $f \in \mathcal{F}(M) = \Lambda_0(M)$ it should be the usual directional derivative:

$$L_\xi : \Lambda_0(M) \to \Lambda_0(M)$$

$$L_\xi : \quad f \to \xi(f) \qquad \left(= \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i} f(x) \right)$$

- **Recall:** once we define L_ξ on Λ_0 and a basis of $\Lambda_1(M)$, then by linearity and the Leibniz rule, L_ξ will automatically be defined on all of $\Lambda(M)$.

- Consider, therefore, any $df \in \Lambda_1(M)$, e.g., the basis vectors $df = dx^i$.

recall that df is the gradient vector field of the function f.

- Then it is natural to define the directional derivative of a gradient field of a function to be the gradient of the directional derivative of the function: (because derivatives ought to commute and the gradient is a derivative too.)

$$L_\xi : \Lambda_1(M) \to \Lambda_1(M)$$

$$L_\xi : \quad df \to \underbrace{d(\underbrace{\xi(f)}_{\in \Lambda_0(M)})}_{\in \Lambda_1(M)}$$

i.e.: $$\underbrace{L_\xi(df)}_{\text{directional derivative of gradient}} = \underbrace{d(\xi(f))}_{\text{gradient of directional derivative}} \qquad \textbf{(D)}$$

Question: Now that L_ξ is a fully defined derivation

$$L_\xi : \Lambda(M) \to \Lambda(M),$$

can we relate it to d and i_ξ? **Yes:**

Cartan's equation:

$$\boxed{L_\xi = \overbrace{d \circ i_\xi + i_\xi \circ d}^{\text{Exercise: show it is a derivation}}}$$

Proof:

check on $\Lambda_0(M)$: $\quad \underset{}{L_\xi} \overset{\Lambda_0(M) \ni}{f} = d \circ \underbrace{i_\xi(f)}_{\substack{=0 \text{ because} \\ f \in \Lambda_0(M)}} + i_\xi(df) = 0 + df(\xi) = \xi(f)$ ✓

check on basis of $\Lambda_1(M)$, e.g. $df = dx^i$: $\quad L_\xi df = d \circ \overbrace{i_\xi(df)}^{= df(\xi) = \xi(f)} + i_\xi \circ \overbrace{ddf}^{\substack{0 \\ \|}} = d(\xi(f))$ ✓ (because: $d^2 = 0$)

I.e., indeed, as in (D): directional derivative of gradient = gradient of directional derivative

Definition:

For any linear maps $A : \Lambda(M) \to \Lambda(M)$, $B : \Lambda(M) \to \Lambda(M)$ we define their commutator (or Lie-, or Poisson bracket):

$$[A, B] := A \circ B - B \circ A$$

Examples of maps:

$$d : \Lambda(M) \to \Lambda(M)$$

$$i_\xi : \Lambda(M) \to \Lambda(M)$$

$$L_\xi : \Lambda(M) \to \Lambda(M)$$

For the commutators of d, i_ξ and L_ξ one can prove:

Proposition:

- $[L_\xi, d] = 0$
- $[L_{\xi_1}, L_{\xi_2}] = L_{[\xi_1, \xi_2]}$
- $[L_{\xi_1}, i_{\xi_2}] = i_{[\xi_1, \xi_2]}$

Exercise: prove this

Here we used on the right hand side that also vector fields

$$\xi : \Lambda_0(M) \to \Lambda_0(M),$$

have commutators:

$$[\xi, \eta](f) = \xi(\eta(f)) - \eta(\xi(f)) = \sum_{i,j=1}^{m} \underbrace{\left(\xi^i \frac{\partial}{\partial x^i} \eta^j \frac{\partial}{\partial x^j} f - \eta^i \frac{\partial}{\partial x^i} \xi^j \frac{\partial}{\partial x^j} f \right)}$$

The terms with the second derivatives cancel because: $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} f$

$$= \sum_{i,j=1}^{m} \left(\underbrace{\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}}_{=: \nu^j} \right) \frac{\partial}{\partial x^j} f$$

$$= \sum_{j=1}^{m} \nu^j \frac{\partial}{\partial x^j} f = \nu(f)$$

Questions:

Since L_ξ is the directional derivative on $\Lambda(M)$:

- Can L_ξ be extended to a directional derivative for all tensor fields? Yes!

- Can L_ξ be expressed as a Newton-Leibniz limit similar to

need an analog: a shift on a manifold, in the direction given by ξ.

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(\overbrace{x + \varepsilon}) - f(x)}{\varepsilon}$$

? Yes!

To this end:

The geometric definition of L_ξ:

- Recall that for any path

$$\gamma : \mathbb{R} \supset J \to M$$
$$\gamma : \quad t \mapsto \gamma(t)$$

an open interval of $\mathbb{R}$

we have a tangent vector $\bar{\gamma}(t) \in T_{\gamma(t_0)}(M)$ at each point $\gamma(t)$ of the path:

$$\bar{\gamma}(t) : f \to \bar{\gamma}(t)(f) = \frac{d}{dt} f(\gamma(t))\Big|_{t=t_0}$$

(the geom. definition of the tangent space)

- Definition: For a given vector field, ξ, a path γ is called an integral curve of ξ, if

$$\bar{\gamma}(t) = \xi(\gamma(t))$$

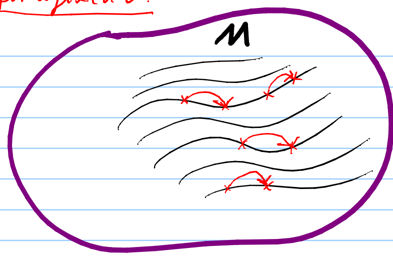
path's velocity vector at $\gamma(t)$ — vector of field ξ at $\gamma(t) \in M$.

- From theory of ODEs:

For every $p \in M$ there exists a maximal (i.e. inextendible) C^∞ integral curve through p.

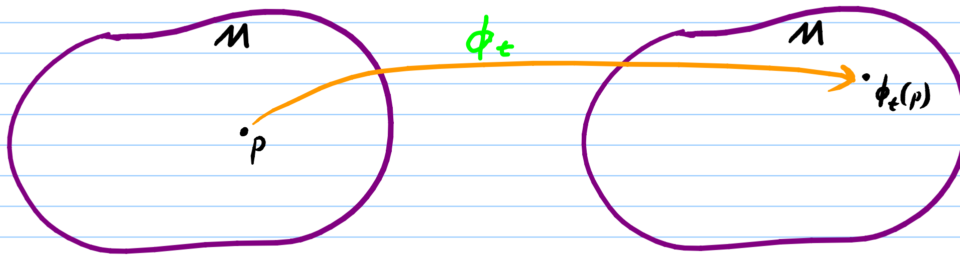
- Thus, ξ yields a "flow": (at least for small t, locally):

for a fixed t:



i.e., for any fixed value of the flow parameter t each point of M is mapped into another point of M.

- The flow is a diffeomorphism "$\phi_t : M \to M$":



- As always, a diffeomorphism of manifolds induces corresponding isomorphisms of the tangent, cotangent and all tensor spaces at p and at $\phi_t(p)$ respectively:

$$\phi_t^* : T_p(M)^r_s \to T_{\phi_t(p)}(M)^r_s$$

- Recall: A tensor field τ assigns to each $p \in M$ a tensor $\tau(p) \in T_p(M)^r_s$.

Definition:

We say that a tensor field τ is invariant under the flow induced by the vector field ξ if:

(The flow produces an image of M in M:)

$$\underbrace{\phi_t^*(\tau(p))}_{\text{image of the tensor field's value at } p} = \underbrace{\tau(\phi_t(p))}_{\text{tensor field's value at the image of } p} \qquad \forall t\ \forall p$$

Definition:

geom. definition

The Lie derivative of any tensor field τ at the point $p = \gamma(0) \in M$ with respect to the flow induced by a vector field ξ is defined through:

$$L_\xi \tau := \lim_{t\to 0} \frac{1}{t}\left(\phi_t^{*-1}(\tau) - \tau\right)$$

$$\text{i.e.}\quad L_\xi(\tau)(p) = \lim_{t\to 0}\frac{1}{t}\Big[\underbrace{(\phi^*)^{-1}\big(\overbrace{\tau(\gamma(t))}^{\text{tensor field value at image of } p,\ \text{i.e. } \in T_{\gamma(t)}(M)^r_s}\big)}_{\in T_p(M)^r_s} - \tau(\underbrace{p}_{=\gamma(0)})\Big]$$

Explicitly, in a chart:

- $\phi: x \to \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t\xi^i(x) + O(t^2)$
- Jacobian matrix: $\dfrac{\partial \tilde{x}^i}{\partial x^j} = \delta^i_j + t\underbrace{\dfrac{\partial \xi^i(x)}{\partial x^j}}_{\text{we write } = \xi^i_{,j}} + O(t^2)$
- Inverse Jacobian: $\dfrac{\partial x^i}{\partial \tilde{x}^j} = \delta^i_j - t\dfrac{\partial \xi^i(x)}{\partial x^j} + O(t^2)$
- Image of tensor at $\tau(\tilde{x})^{i_1\ldots i_r}_{j_1\ldots j_s}$ under flow, backwards, $\tilde{x}\to x$, has the

From now, we will omit writing Σ: Twice occurring indices are always to be summed over (Einstein convention)

components:

$$\phi^{*-1}(\tau(\tilde{x}))^{i_1\ldots i_r}_{j_1\ldots j_s} = \tau^{\tilde{i}_1\ldots\tilde{i}_r}_{\tilde{j}_1\ldots\tilde{j}_s}(\tilde{x})\,\frac{\partial x^{i_1}}{\partial \tilde{x}^{\tilde{i}_1}}\cdots\frac{\partial x^{i_r}}{\partial \tilde{x}^{\tilde{i}_r}}\,\frac{\partial \tilde{x}^{\tilde{j}_1}}{\partial x^{j_1}}\cdots\frac{\partial \tilde{x}^{\tilde{j}_s}}{\partial x^{j_s}}$$

$$= \tau^{\tilde{i}_1\ldots\tilde{i}_r}_{\tilde{j}_1\ldots\tilde{j}_s}(x+t\xi)\left(\delta^{i_1}_{\tilde{i}_1} - t\xi^{i_1}_{,\tilde{i}_1}\right)\cdots\left(\delta^{i_r}_{\tilde{i}_r} - t\xi^{i_r}_{,\tilde{i}_r}\right)$$

$$\cdot\left(\delta^{\tilde{j}_1}_{j_1} + t\xi^{\tilde{j}_1}_{,j_1}\right)\cdots\left(\delta^{\tilde{j}_s}_{j_s} + t\xi^{\tilde{j}_s}_{,j_s}\right) + O(t^2)$$

$$f_{,k} := \frac{\partial}{\partial x^k} f$$

$$= \tau^{i_1\dots i_r}_{j_1\dots j_s}(x) + t\,\tau^{i_1\dots i_r}_{j_1\dots j_s\,,k}(x)\,\xi^k(x)$$

$$-t\,\tau^{\bar{i}\dots i_r}_{j_1\dots j_s}(x)\,\xi^{i_1}_{,\bar{i}}(x) - \dots - t\,\tau^{i_1\dots \bar{i}}_{j_1\dots j_s}(x)\,\xi^{i_r}_{,\bar{i}}(x)$$

$$+t\,\tau^{i_1\dots i_r}_{\bar{j}\dots j_s}(x)\,\xi^{\bar{j}}_{,j_1}(x) + \dots + t\,\tau^{i_1\dots i_r}_{j_1\dots \bar{j}}(x)\,\xi^{\bar{j}}_{,j_s}(x)$$

$$\Rightarrow (L_\xi \tau)^{i_1\dots i_r}_{j_1\dots j_s}(x) = \lim_{t\to 0}\frac{1}{t}\left(\phi^{*-1}(\tau(x))^{i_1\dots i_r}_{j_1\dots j_s} - \tau^{i_1\dots i_r}_{j_1\dots j_s}(\overbrace{x(0)}^{x})\right)$$

$$= \tau^{i_1\dots i_r}_{j_1\dots j_s,k}(x)\,\xi^k(x) - \tau^{\bar{i}\dots i_r}_{j_1\dots j_s}(x)\,\xi^{i_1}_{,\bar{i}}(x) - \dots$$

$$+ \tau^{i_1\dots i_r}_{\bar{j}\dots j_s}(x)\,\xi^{\bar{j}}_{,j_1}(x) + \dots\ \tau^{i_1\dots i_r}_{j_1\dots \bar{j}}(x)\,\xi^{\bar{j}}_{,j_s}(x)$$

- Equivalent to algebraic definition of L_ξ?

  Yes: Check, e.g., that action on $\Lambda_0(M)$ and $\Lambda_1(M)$ is the same:

  - For $\tau \in \Lambda_0(M)$ we have $\tau = \tau(x)$

    $L_\xi \tau(x) = \xi^k \tau_{,k} = \xi^k \frac{\partial}{\partial x^k}\tau(x)$ is gradient ✓

  - Co-Vector field: $\tau = \tau_j(x)\,dx^j \in \Lambda_1(M)$

    $L_\xi \tau(x) = \left(\xi^k(x)\,\tau_{j,k}(x) + \tau_k(x)\,\xi^k_{,j}(x)\right) dx^j$

    Exercise: verify that this agrees with the algebraically defined action of L_ξ on $\Lambda_1(M)$.

Collected properties: (without proof)

- $L_\xi : T_p(M)^r_s \to T_p(M)^r_s$ (i.e. not just $\Lambda_s \to \Lambda_s$)

- In particular, the Lie derivative of a vector field η is:

$$L_\xi : \eta \to L_\xi(\eta) = [\xi, \eta]$$

- One also finds:

$$L_{\xi+\eta} = L_\xi + L_\eta$$

$$L_{[\xi,\eta]} = [L_\xi, L_\eta] \quad \left(= L_\xi \circ L_\eta - L_\eta \circ L_\xi \right)$$

- Does it still obey a Leibniz rule?

Yes: $$L_\xi(\tau \otimes \sigma) = L_\xi(\tau) \otimes \sigma + \tau \otimes L_\xi(\sigma)$$

(tensors form an algebra w. respect to multiplication $\otimes$)