GR for Cosmology, Achim Kempf — Lecture 7

Recall: Physical motivation for the "Metric Tensor"

- In Minkowski space, in inertial and cartesian coordinates:

4-dim space-time distance!

$$\left[\text{distance}(x,\hat{x})\right]^2 = -(x^0-\hat{x}^0)^2+(x^1-\hat{x}^1)^2+(x^2-\hat{x}^2)^2+(x^3-\hat{x}^3)^2$$

indep. of choice of inertial cds.

$$= \eta_{\mu\nu}(x^\mu-\hat{x}^\mu)(x^\nu-\hat{x}^\nu)$$

with $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

- In Minkowski space, in an arbitrary coordinate system:

$$\left[\text{distance}(x,\hat{x})\right]^2 = g_{\mu\nu}(x)(x^\mu-\hat{x}^\mu)(x^\nu-\hat{x}^\nu) + \mathcal{O}^3$$

(e.g. polar cds, or accelerated cds)

with $g_{\mu\nu}(x) \neq \eta_{\mu\nu}$

complicated higher order terms

- Generalization to curved space-time, historically:

Allow even such $g_{\mu\nu}(x)$ which in no coordinate system obey:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \quad \text{for all } x \in \mathcal{M}$$

$\Rightarrow$ $g_{\mu\nu}(x)$ is not simply $\eta_{\mu\nu}$ in noninertial coordinates

$\Rightarrow$ Such $g_{\mu\nu}(x)$ take us beyond special relativity!

- Enforce Einstein's equivalence principle:

Require $g_{\mu\nu}$ to be such that for each $x \in \mathcal{M}$ there exists a coordinate system so that at least at x:

$$g_{\mu\nu}(x) = \eta_{\mu\nu}$$

(i.e., locally, special relativity holds
$\text{dist}(x,\hat{x})^2 = \eta_{\mu\nu}(x-\hat{x})^\mu(x-\hat{x})^\nu + \mathcal{O}^3$
to lowest nontrivial order.)

(Recall equiv. principles (EP):

If freely falling small masses
fall equally ⇒ "weak EP"
+ same internal non-grav. physics ⇒ "Einst. EP"
+ same internal grav. physics ⇒ "strong EP")

Recall: Math. definition of the metric tensor:

- g is covariant tensor of rank $(0,2)$
  (because η is in special relativity)
- Thus, if n cotangent vector fields $\theta^\mu(x)$ (e.g. $\theta^\mu(x) = dx^\mu$) form bases at each point x, then g is of the form:

$$g(x) = g_{\mu\nu}(x)\, \theta^\mu(x) \otimes \theta^\nu(x)$$

recall: $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ and $g_{\mu\nu}$ is invertible (since nondegenerate)

- $g_{\mu\nu}(x)$ invertible $\Rightarrow$ there exists a tensor g^{-1} of rank $(2,0)$:

$$g^{-1}(x) = g^{\mu\nu}(x)\, \overset{\text{dual basis}}{e_\mu}(x) \otimes e_\nu(x) \text{ with } g^{\mu\nu}(x)\, g_{\nu\rho}(x) = \delta^\mu_\rho$$

⇝ Modern view of the Einsteinian equivalence principle:

Recall: We asked that for each point $p \in M$ there is a coordinate system so that, when choosing the bases $\{dx^i\}, \{\frac{\partial}{\partial x^j}\}$

then $g(x) = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu$, $g_{\mu\nu}(x) = g(x)\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$

obeys: $g_{\mu\nu}(p) = \eta_{\mu\nu}$ (in general only at p)

Modern formulation of the Einsteinian equivalence principle:

Independently of any choice of coordinate system:

There are choices of dual bases $\{\theta^\mu(x)\}, \{e_\nu(x)\}$ of $T_x(M), T_x(M)^*$,

so that: $g_{\mu\nu}(x) = g\left(e_\mu(x), e_\nu(x)\right) = \eta_{\mu\nu}$ $\quad \forall x \in M$

Now, knowing distances through $g_{\mu\nu}$, what else follows?

- Distances yield volumes, namely $g_{\mu\nu}(x)$ induces an $\Omega(x)$.
- g, g^{-1} yield duality of covariance and contravariance.
- g yields "Hodge star" $*: \Lambda_p \to \Lambda_{n-p}$ duality.
- $*$ yields $(\,,\,)$ making the Λ_p Hilbert spaces. for Riemannian manifolds
- g yields co-derivative $\delta: \Lambda_p \to \Lambda_{p-1}$
- d, δ yield the Laplacian/d'Alembertian $\Delta: \Lambda_p \to \Lambda_p$

  $\rightsquigarrow$ We can formulate wave equations on M!

Proposition:

Given a notion of distance, i.e., a metric, g, this also induces a volume form Ω.

(i.e., a positive $\Omega \in \Lambda_n(M)$, i.e., that when integrated over any portion of M yields a positive number)

Namely:

- Assume, as always, that M is oriented.
- Consider a positive chart.

  (i.e. has positive det(Jacobian) with given atlas)

Then:

$:= |\det(g_{ij}(x))|$

$$\Omega := \sqrt{|g|}\; dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

is a well-defined volume form.

Proof:

- Nonzero for all $p \in M$?
  Yes, because g is assumed non-degenerate.
- Well-defined, i.e., is definition chart-independent?
  Yes: To see this, change chart: $x \to \tilde{x}$

Then: $\tilde{g}_{ij}(\tilde{x}(x)) = g_{rs}(x) \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}$ because covariant

i.e., as matrices:

$$\tilde{g} = \left(\frac{\partial x}{\partial \tilde{x}}\right)^t g \left(\frac{\partial x}{\partial \tilde{x}}\right)$$ now take determinant:

$$\Rightarrow \quad |\tilde{g}| = \left|\frac{\partial x}{\partial \tilde{x}}\right|^2 |g|$$ i.e. $|\tilde{g}|^{1/2} = \left|\frac{\partial x}{\partial \tilde{x}}\right| |g|^{1/2}$

Also: $d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n = \det\left(\frac{\partial \tilde{x}}{\partial x}\right) dx^1 \wedge \dots \wedge dx^n$

$$\Rightarrow \quad |\tilde{g}|^{1/2} d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n = \underbrace{\left|\frac{\partial \tilde{x}}{\partial x}\right| \left|\frac{\partial x}{\partial \tilde{x}}\right|}_{= 1} |g|^{1/2} dx^1 \wedge \dots \wedge dx^n \quad \checkmark$$

Notation: (Ω is an n-form. What are its coefficients, as a covariant $(0,n)$ tensor?)

- Define:

$$\varepsilon_{i_1,\dots,i_n} := \begin{cases} +1 & \text{if } (i_1,\dots,i_n) \text{ is even permutation of } (1,2,\dots,n) \\ -1 & \text{if } (i_1,\dots,i_n) \text{ is odd permutation of } (1,2,\dots,n) \\ 0 & \text{else} \end{cases}$$

unlike in SRT, $\varepsilon_{\dots}$ is not canonical, because Ω is:

- Then, Ω also reads:

$$\Omega = \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^n \qquad (n\text{-form})$$

$$= \underbrace{\sqrt{|g|}\, \varepsilon_{i_1,\dots,i_n}}_{=: \Omega_{i_1,\dots,i_n}} dx^{i_1} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_n}$$

$$\Omega = \Omega_{i_1 \dots i_n} dx^{i_1} \otimes \dots \otimes dx^{i_n} \qquad (\text{covariant tensor})$$

- Ω is called the "canonical", or "(pseudo)Riemannian", or "metric", volume form.

Q: Other use of g ?

A: One needs g to formulate d'Alembertian Δ, or $\Box$, for wave equations.

Why? a.) $\Box$ should be non-directional 2^{nd} derivative, but $d^2=0$.

b.) need e.g. $\Box: \Lambda^0 \to \Lambda^0$ for Klein Gordon, i.e, need degree of forms conserved by $\Box$.

Strategy:

A) Use g for a covariant $\leftrightarrow$ contravariant tensors relation

B) Define a map "Hodge$*$" : $\Lambda_r \to \Lambda_{n-r}$

C) Define the "Coderivative": $\delta : \Lambda_r \to \Lambda_{r-1}$

D) Define "Laplacian/d'Alembertian": $\Box := d\delta + \delta d$

Then, e.g., the Klein Gordon equation reads:

$$(\Box + m^2)\,\phi = 0$$

A) Covariant $\leftrightarrow$ contravariant tensors equivalence through g :

- $g(x)$ can be used as a map: by evaluation of one tensor factor:

$$g(x): \quad T_x(M)' \to T_x(M),$$

$$\theta^\nu(e_i) = \delta^\nu_i$$

$$g(x): \xi^i(x) e_i(x) \to g_{\mu\nu}(x)\, \theta^\mu(x) \otimes \theta^\nu(x) \left(\xi^i(x) e_i(x)\right)$$

$$= \underbrace{g_{\mu\nu}(x)\xi^\nu(x)}_{\in \mathcal{F}_x(M)} \underbrace{\theta^\mu(x)}_{\in T_x(M),} \quad \in T_x(M)',$$

$\Rightarrow$ For the coefficient functions we have: $g \;:\; \xi^\mu(x) \to \omega_\nu(x) = g_{\nu\rho}(x)\xi^\rho(x)$ (relative to bases θ^i, e_j)

- Conversely, g^{-1} acts as:

$$g^{-1}(x): T_x(M)_{,} \to T_x(M)'$$

$$g^{-1}(x): \quad \omega_\mu(x) \to \xi^\nu(x) = g^{\nu\rho}(x)\,\omega_\rho(x)$$

- In this way, g, g^{-1} can lower or raise any tensor index, e.g.: $g: t^{ij}{}_k \to t_i{}^j{}_k = g_{ij}\, t^{ij}{}_k$

  and: $g^{-1}: \tau^{ij}{}_k \to \tau^{ijk} = g^{jk}\,\tau^{ij}{}_k$

B) The Hodge $*$ map: $\Lambda_p \to \Lambda_{n-p}$

(Recall: $\dim(\Lambda_p) = \binom{n}{p} = \binom{n}{n-p} = \dim(\Lambda_{n-p})$)

Idea:

- each $v \in \Lambda_p$ is a covariant tensor
- through g it is equivalent to a contravariant tensor $\tilde{v}$
- can feed Ω with $\tilde{v}$ to obtain $*v \in \Lambda_{n-p}$.

Concretely:

anything totally antisymmetric

one could choose other bases.

- Consider any $v := \frac{1}{p!} v_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \Lambda_p$

convenient because drops out here

coefficients as a covariant tensor

$$= v_{i_1 \dots i_p} dx^{i_1} \otimes \dots \otimes dx^{i_p}$$

- Use g^{-1} to define a contravariant image of v:

$$\tilde{v} = \tilde{v}^{i_1 \dots i_n} \frac{\partial}{\partial x^{i_1}} \otimes \dots \frac{\partial}{\partial x^{i_p}}$$

where $\tilde{v}^{i_1, \dots, i_p} := g^{i_1 j_1} g^{i_2 j_2} \dots g^{i_p j_p} v_{j_1 \dots j_p}$

- Apply Ω on $\tilde{v}$:

$$\Omega(\tilde{v}) = \underbrace{\Omega_{i_1 \dots i_n} \tilde{v}^{i_1 \dots i_p}}_{\overset{!!}{=} (*v)_{i_{p+1}, i_{p+2}, \dots, i_n}} \overbrace{dx^{i_{p+1}} \otimes \dots \otimes dx^{i_n}}^{n-p \text{ factors}} \in \Lambda_{n-p}$$

- Define $*v := \Omega(\tilde{v})$, i.e.:

$$*v = (*v)_{i_1, \dots, i_{n-p}} dx^{i_1} \otimes \dots \otimes dx^{i_{n-p}}$$

$$*v = \frac{1}{(n-p)!} (*v)_{i_1 \dots i_{n-p}} dx^{i_1} \wedge \dots \wedge dx^{i_{n-p}}$$

Proposition:

Assume $v \in \Lambda_p$. Then

$$**v = (-1)^{p(n-p)+s} v$$

E.g. $s=1$ for space-time

What is s? The "signature" of g is $\mathrm{sgn}(g) = (r,s)$, where in diagonal form: $g = \begin{pmatrix} \left.\begin{matrix}1 & \\ & 1\end{matrix}\right\}r & & \\ & \left.\begin{matrix}-1 & \\ & -1\end{matrix}\right\}s & \\ & & \ddots \end{pmatrix}$

Use $*$ to turn $\Lambda(M)$ into an "Inner Product Space":

Definition: The Hodge $*$ provides a "scalar" (or also called "inner") product for $\Lambda(M)$:

$$\boxed{(\underset{\substack{\uparrow\\ \text{p-form}}}{\alpha}, \underset{\substack{\uparrow\\ \text{p-form}}}{\beta}) := \int_M \underbrace{\alpha \wedge \overbrace{*\beta}^{(n-p)\text{ form}}}_{n\text{ form}}}$$

Exercises:

1.) Write (α,β) in coordinates

2.) Show that $(,)$ is always positive definite on Λ_0, i.e., $(\alpha,\alpha) > 0 \ \forall \alpha \in \Lambda_0, \alpha \neq 0$.

This definition is extended linearly to forms that are lin. comb. of forms of arb. degree, p.

Notes:

- If g is indefinite, then also $(,)$ is indefinite.
- If g is positive definite, i.e., if M is Riemannian, then $(,)$ is positive definite and Λ becomes a Hilbert space.

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c) $(,)$ yields an adjoint for d, the Co-derivative δ:

Recall: For any operator $A: D_A \subset \mathcal{H} \to \mathcal{H}$ (with D_A dense, i.e., $\bar{D}_A = \mathcal{H}$), its adjoint A^+ is defined to have the domain

$$D_{A^+} := \{ v \in \mathcal{H} \,/\, \exists w \in \mathcal{H} \ \ \forall z \in D_A : \ \langle v, Az \rangle = \langle w, z \rangle \}$$

and this action: $A^+ v := w$. We then have:

$$\langle A^+ v, z \rangle = \langle v, Az \rangle \quad \forall z \in D_A, v \in D_{A^+}$$

Definition:

The co-derivative, δ, is the (anti-) adjoint of d with respect to the inner product $(,)$ on $\Lambda(M)$:

$$\boxed{(\delta\alpha, \beta) := -(\alpha, d\beta)} \qquad \forall \alpha \in D_\delta, \beta \in D_d$$

c) The Codifferential δ explicitly

Clearly: $\delta : \Lambda^p(M) \to \Lambda^{p-1}(M)$

Proposition: $\delta : v \to \left((-1)^{np+n+s} * d *\right) v$

(Some authors define δ as the negative of this)

Properties:

- $\delta^2 = 0$
- In coordinates:
  $$(\delta\omega)^{i_1 \dots i_{p-1}} = \frac{1}{\sqrt{|g|}}\left(\sqrt{g}\, \omega^{k\, i_1 \dots i_{p-1}}\right)_{,k}$$
- If M is contractible (and in every contractible part):
  $$\delta v = 0 \Rightarrow \exists \omega : \ v = \delta\omega$$

Exercises:

- Show the above.
- Determine whether or not δ is a derivation.

Use d and δ to obtain the Maxwell equations on M

- Define:

electric field

"Field strength": $$F_{\mu\nu}(x) := \begin{pmatrix} 0, & -E_1, & -E_2, & -E_3 \\ E_1, & 0, & B_3, & -B_2 \\ E_2, & -B_3, & 0, & B_1, \\ E_3, & B_2, & -B_1, & 0, \end{pmatrix}, \quad F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

magn. field

"Field strength" 2-form

"Current" 3-form → $$j(x) := \frac{1}{3!}\varepsilon_{\mu\nu\lambda\rho}\, j^\mu\, dx^\nu \wedge dx^\lambda \wedge dx^\rho$$

"current 4-vector"

- Then: The Maxwell Eqns read:

"Homogeneous Maxwell equations" (indep. of metric)

$$\boxed{dF = 0, \quad \delta F = *j}$$

"Inhomogeneous Maxwell equations" (dependent on the metric)

Current 1-form, i.e., cotangent vector field

Remarks:

- F is assumed to be an exact 2-form, i.e.,:

$$F = dA$$

(the 1-form A is called the 4-potential)

- This already implies the homogenous Maxwell equations:

$$dF = d^2A = 0 \quad !$$

⇝ One calls them "structure equations".

- General relativity also possesses structure equations.

Remark:

The gauge principle of electrodynamics is the observation that, for any $\omega \in \Lambda_0$:

$$A \quad \text{and} \quad \tilde{A} := A + d\omega$$

describe the same physics.

The Aharonov-Bohm effect and topological phases in general, can make A itself visible when Poincaré lemma doesn't apply.

They do because the (classically) observable fields are only the E and B fields in F and since $d^2 = 0$:

$$\text{observable } E \text{ and } B \text{ fields} \longrightarrow F = dA = d\tilde{A}$$

D The Laplacian/d'Alembertian, Δ, $\Box$:

- Definition of the Laplacian:

$$\boxed{\Delta := \delta d + d\delta}$$

Some authors define Δ as the negative of this and let δ be the adjoint of d.

- Clear: $\Delta : \Lambda^p(M) \to \Lambda^p(M)$
- If signature $s=1$: Then also called d'Alembertian and denoted $\Box := d\delta + \delta d$.
- Action on, e.g., $f \in \Lambda_0(M)$ in a chart:

Exercise: verify

$$\Box f = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\, g^{\mu\nu} f_{,\mu}\right)_{,\nu}$$

$$\left(= \left(-\frac{\partial^2}{\partial x^{0^2}} + \frac{\partial^2}{\partial x^{1^2}} + \frac{\partial^2}{\partial x^{2^2}} + \frac{\partial^2}{\partial x^{3^2}}\right) f \quad \text{if } g = \eta \right)$$

, if Λ is a Hilbert space

Properties of the d'Alembertian, $\Box$ in the Hilbert space $\Lambda(M)$:

* **Defined:**

$$\Box : \Lambda_r(M) \to \Lambda_r(M)$$

$$\Box : \nu \to (\delta d + d\delta)\nu$$

* In the Hilbert space $\Lambda(M)$:

$\Box = \delta d + d\delta$ obeys $(\alpha, \Box\beta) = (\Box\alpha, \beta)$

* $\Box$ is self-adjoint, $\Box = \Box^+$, for suitable boundary conditions, or if $\partial M = \phi$, and assuming $(\ ,\ )$ is positive definite.

* Exercises:
  - Verify $\Box = \Box^+$ formally, using only $\delta = -d^+$.
  - Verify that $\Box * = * \Box$, $\Box d = d \Box$, $\Box\delta = \delta\Box$.

Consequences of the self-adjointness of $\Box$: – if Λ is a Hilbert space

A) The operators Δ and $\Box$ can be diagonalized, with real spectrum.

B) For Riemannian manifolds, $\mathrm{spec}(\Delta) \subset [0,\infty)$.

C) For compact Riem. manifolds (of finite volume): $\mathrm{spec}(\Delta)$ is discrete.

Still the finite volume Riemannian case.

D) Then, $\mathrm{spec}(\Delta)$ is carrying a lot of information about (M,g)!

Remark: There exists a related mathematical discipline, called "Spectral Geometry", combining differential geometry with functional analysis, i.e., the languages of general relativity and quantum theory!

of Riemannian manifolds →

Application: Klein-Gordon "action":

$$\overbrace{g^{\gamma\delta}\, e_\gamma \otimes e_\delta}^{\in T^2}\,\big(\overbrace{\phi_{,\alpha}\theta^\alpha}^{\in T_1} \otimes \overbrace{\phi_{,\beta}\theta^\beta}^{\in T_1}\big)$$

$$\underset{\substack{\uparrow\\ \text{Klein Gordon field } \phi\in\mathcal{F}(M)}}{S[\phi]} := \frac{1}{2}\int_M \overbrace{\underset{\substack{\Uparrow\\ \Lambda_0}}{g^{\mu\nu}}\ \underset{\substack{\Uparrow\\ \Lambda_0}}{\phi_{,\mu}}\ \underset{\substack{\Uparrow\\ \Lambda_0}}{\phi_{,\nu}}}^{=}\ \underset{\substack{\Uparrow\\ \Lambda_n}}{\Omega}$$

(Recall special relativity: $S[\phi] = \int_{\mathbb{R}^4} \eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\, d^4x$)

$$= \frac{1}{2}\int_M g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi\right)\left(\frac{\partial}{\partial x^\nu}\phi\right)\overbrace{\sqrt{|g(x)|}\, d^n x}^{\overset{\Omega}{=}}$$

next: integrate by parts!

$$= \frac{1}{2}\int_M -\phi\ \underbrace{\frac{\partial}{\partial x^\nu}\left(\sqrt{|g|}\, g^{\mu\nu}\frac{\partial}{\partial x^\mu}\phi\right)\frac{1}{\sqrt{|g|}}}_{=\Box\phi}\ \underbrace{\sqrt{|g|}\, d^n x}_{=\Omega}$$

$$= -\frac{1}{2}\int \phi(\Box\phi)\,\Omega$$

Obtain the Klein Gordon wave equation:

- Recall: Euler Lagrange equation $\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}$
- Here: $\mathcal{L} = -\frac{1}{2} \phi \Box \phi$ (the 0-form that we are integrating: $S' = \int_M \mathcal{L}\, \Omega$)
- Obtain Klein Gordon equation:

$$\Box \phi = 0$$

(with "mass": $\mathcal{L} = -\frac{1}{2} \phi (\Box + m^2) \phi$
yielding $(\Box + m^2) \phi = 0$)

Q: Which physical fields are described by K-G fields?

there are many sorts of mesons. Most important mesons: "Pions". They transmit the nuclear force among protons & neutrons

A:
- Meson fields
- Higgs field (Gives all particles their mass. Found at LHC. Nobel to Higgs, Englert (Brout) in 2013)
- Inflaton field (crucial ingredient in modern cosmology → see later)