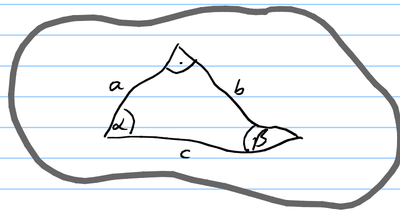
How to describe the "shape" of a manifold?

Historically:

E.g., on a potato-shaped surface:



$$a^2 + b^2 \neq c^2$$

$$\alpha + \beta + 90^\circ \neq 180^\circ$$

Helmholtz & Gauß already considered checking for curvature of space this way.

Recall:

Defined $g_{\mu\nu}(x)$

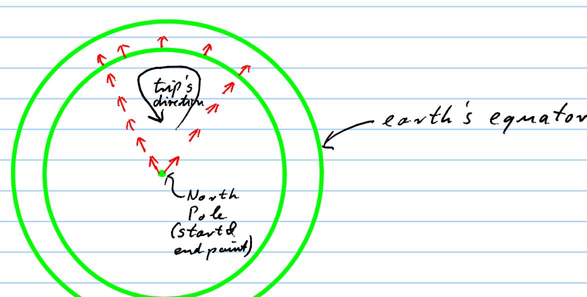
$\Rightarrow$ infinitesimal distances $\Rightarrow$ finite distances $\Rightarrow$ shape

Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the nontriviality of the parallel transport of vectors on the manifold:

Example:

- start with a vector at North Pole.
- parallel transport down to some lower latitude, along that latitude and then back to pole.
- vector will arrive at pole rotated, in spite of having only been parallel transported!



This motivates:

yep: another derivative!

Try to define local shape through "derivative" of vectors with respect to parallel transport!

Recall: Lie derivative insensitive to g. Indeed, for $\xi_1 = \frac{\partial}{\partial x^1}$, $\xi_2 = \frac{\partial}{\partial x^2}$, we have $[L_{\xi_1}, L_{\xi_2}] = L_{[\xi_1,\xi_2]} = 0 \Rightarrow$ No shape info from L_ξ!

The Covariant Differentiation, ∇:

Definition: Any linear map of tangent vector fields

$$\nabla : T^1(M) \times T^1(M) \to T^1(M)$$

$$\nabla : \quad \eta, \xi \to \nabla_\xi \eta$$

obeying

(I) $\quad \nabla_{f\xi} \eta = f \nabla_\xi \eta, \ \forall f \in \mathcal{F}(M)$

(II) $\quad \nabla_\xi (f\eta) = \overbrace{\xi(f)}^{\nabla_\xi f =} \eta + f \nabla_\xi \eta$ (Leibniz rule)

is called a covariant derivative or affine connection.

Note:

For now, let us assume a metric has not (yet) been specified, so we are free to choose ∇, and this choice defines the shape of M!

∇ in a chart:

- Choose as bases for $T_x(M)$, e.g.: $\left\{\frac{\partial}{\partial x^i}\right\}$
- Given a covariant derivative ∇, consider its action on basis vectors, such as, e.g.,: $\xi = \frac{\partial}{\partial x^i}, \eta = \frac{\partial}{\partial x^j}$:

Recall: $L_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$

$$\boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k{}_{ij}(x) \frac{\partial}{\partial x^k}}$$

The Γ^k_{ij} are called "Christoffel symbol" or "Connection coefficients".

Thus, via the axioms:

$$\nabla_\xi \eta = \nabla_{\xi^i \frac{\partial}{\partial x^i}} \left(\eta^j \frac{\partial}{\partial x^j}\right) \overset{(I)}{=} \xi^i \nabla_{\frac{\partial}{\partial x^i}} \left(\underset{\text{function}}{\eta^j} \underset{\text{vector}}{\frac{\partial}{\partial x^j}}\right)$$

$\leftarrow$ recall: $\nabla_\alpha f = \alpha(f) \ \forall \alpha \in T_p^1(M), f \in \mathcal{F}_p(M)$

$$\overset{(II)}{=} \xi^i \left(\eta^j{}_{,i} \frac{\partial}{\partial x^j} + \eta^j \Gamma^k{}_{ij}(x) \frac{\partial}{\partial x^k}\right)$$

$$= \left(\xi^i \eta^k{}_{,i} + \xi^i \eta^j \Gamma^k{}_{ij}\right) \frac{\partial}{\partial x^k}$$

Notation: $\quad \eta^k{}_{;i} := \eta^k{}_{,i} + \eta^j \Gamma^k{}_{ij}$

$\uparrow$ semi-colon for covariant derivatives

Thus: $\quad \nabla_\xi \eta = \xi^i \eta^k{}_{;i} \frac{\partial}{\partial x^k} \qquad (*)$

Important: the $\Gamma^k{}_{ij}$ transform non-tensorially when $x \to \bar{x}$:

On one hand:

because $\frac{\partial}{\partial \bar{x}}$ is tangent vector

$$\nabla_{\partial/\partial \bar{x}^a} \frac{\partial}{\partial \bar{x}^b} = \bar{\Gamma}^c{}_{ab} \frac{\partial}{\partial \bar{x}^c} = \bar{\Gamma}^c{}_{ab} \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial}{\partial x^k} \qquad (\mathrm{I})$$

On the other hand:

$$\nabla_{\partial/\partial \bar{x}^a} \frac{\partial}{\partial \bar{x}^b} = \nabla_{\underbrace{\frac{\partial x^i}{\partial \bar{x}^a}}_{\in \mathcal{F}(M)} \frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j} \right) \qquad \text{use axiom (b)} \Rightarrow$$

$$= \frac{\partial x^i}{\partial \bar{x}^a} \nabla_{\frac{\partial}{\partial x^i}} \left(\underbrace{\frac{\partial x^j}{\partial \bar{x}^b}}_{\in \mathcal{F}(M)} \frac{\partial}{\partial x^j} \right) \qquad \text{use Leibniz rule (c)} \Rightarrow$$

$$= \frac{\partial x^i}{\partial \bar{x}^a} \left[\left(\frac{\partial}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k{}_{ij} \frac{\partial}{\partial x^k} \right]$$

$$= \left(\frac{\partial}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k{}_{ij} \frac{\partial}{\partial x^k} \qquad (\mathrm{II})$$

Compare I, II $\Rightarrow$

$$\bar{\Gamma}^c{}_{ab} \frac{\partial x^k}{\partial \bar{x}^c} = \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k{}_{ij} \qquad \left(\cdot \frac{\partial \bar{x}^r}{\partial x^k} \Rightarrow \right)$$

$$\Rightarrow \boxed{\bar{\Gamma}^r{}_{ab} = \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b}} + \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k{}_{ij}}}$$

This term is indep. of Γ
$\Rightarrow$ Γ can be zero in one coordinate system and nonzero in another!

only this term would be there, if the $\Gamma^k{}_{ij}$ were tensor coefficients in the $\frac{\partial}{\partial x^i}$, dx^j bases.

(Can be shown to be equivalent)

Physicists' definition of ∇: Any set of n^3 functions $\Gamma^r_{ab}(x)$ which transform this way are defining a cov. derivative ∇.

The "absolute" covariant derivative ∇:

Consider the covariant derivative but:
without choosing a direction vector ξ:

$$\nabla : T_x(M)^1 \to T_x(M)^1_1$$

$$\nabla : \eta = \eta^i(x)\frac{\partial}{\partial x^i} \to \nabla\eta(x) = \eta^k{}_{;i}(x)\, dx^i \otimes \frac{\partial}{\partial x^k}$$

(I.e. feed the open covariant slot of $\nabla\eta$ with contravariant ξ.)

Indeed: The contraction of $\nabla\eta$ with ξ yields:

$$\nabla\eta(\xi) = \eta^k{}_{;i}\, \underbrace{dx^i(\xi)}_{dx^i(\xi) = \xi(x^i) = \xi^j \frac{\partial}{\partial x^j} x^i = \xi^j \delta^i_j = \xi^i} \frac{\partial}{\partial x^k} = \eta^k{}_{;i}\, \xi^i \frac{\partial}{\partial x^k} = \nabla_\xi \eta \quad \text{ok with } (*)$$

We defined ∇ algebraically. Now, extract the

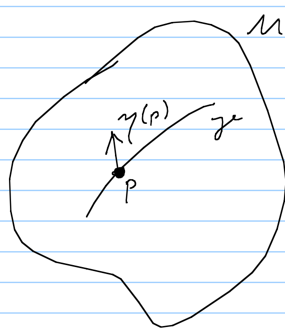
Geometric meaning of ∇:

(If ∇ describes infinitesimal parallel transport it should also describe finite parallel transport)

Definition: Assume ∇ is given. Choose a path $\gamma : \mathbb{R} \to M$.
Then, a tangent vector field η is called auto-parallel along γ, if

$$\nabla_{\dot\gamma}\, \eta = 0$$

i.e. if η doesn't change under parallel transport along the path γ.



Note: We'll see that they always exist, i.e., we can always parallel transport a vector η finite distances.

In a chart,

$$\eta = \eta^i(x)\frac{\partial}{\partial x^i}$$

and

$$\gamma : [a,b] \to M$$

$$\gamma : t \to x^i(t)$$

and the tangent vector:

$$\dot\gamma(x(t)) = \frac{dx^k}{dt}\frac{\partial}{\partial x^k}$$

Thus: $\nabla_{\dot\gamma}\eta = \nabla_{\frac{dx^k}{dt}\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right) = \frac{dx^k}{dt}\nabla_{\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right)$

$$= \frac{dx^k}{dt}\left(\frac{\partial\eta^i}{\partial x^k}\frac{\partial}{\partial x^i} + \eta^i\Gamma^j{}_{ki}\frac{\partial}{\partial x^j}\right)$$

$$= \left(\frac{d\eta^i}{dt} + \frac{dx^k}{dt}\eta^j\Gamma^i{}_{kj}\right)\frac{\partial}{\partial x^i} \overset{!}{=} 0$$

$\Rightarrow$ η autoparallel to γ means:

$$\frac{d\eta^i}{dt} + \eta^j\frac{dx^k}{dt}\Gamma^i{}_{kj} = 0$$

I.e. this is the condition for the vectors of η being parallel translates of each other, along γ.

Conclusion:

This is 1st order ODEs for η. Thus:

Initial condition $\eta(\gamma(0)) \Rightarrow$ solution $\eta(\gamma(t))$ exists at least locally

$\Rightarrow$ Proposition:

Given a path $\gamma : [t,s] \to M$, the autoparallel transport of a tangent vector η at $\gamma(t)$ to $\gamma(s)$ is unique.

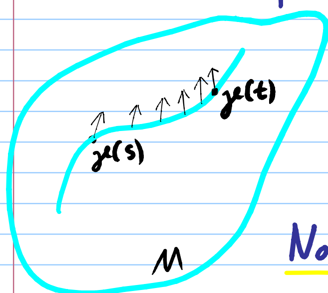
I.e., the path γ defines a parallel transport map τ:

$$\tau(t,s): T'_{\gamma(t)} \to T'_{\gamma(s)}$$

$$\tau(t,s): \eta(\gamma(t)) \to \eta(\gamma(s))$$

- Q: Can one use τ to obtain ∇ as a Newton-Leibniz limit?

- **Proposition:** (for the proof, see e.g. the text by Straumann)



$$\nabla_{\dot\gamma}\, \eta(\gamma(t)) = \frac{d}{ds}\Big|_{s=t} \tau(s,t)\big(\eta(\gamma(s))\big)$$

Note: Since we can choose paths with arbitrary $\dot\gamma$ this equation can be used as a geometric definition of ∇.

∇ for arbitrary tensors:

- The parallel transport map $\tau(s,t)$ transports tangent vectors η from $\gamma(s)$ to $\gamma(t)$.

- **Definition:** $\tau(s,t)$ also parallel transports the dual vectors ω, namely so that contraction is conserved:

$$\boxed{\underbrace{\tau(\omega)}_{\text{parallel transported } \omega}\big(\underbrace{\tau(\eta)}_{\text{parallel transported } \eta}\big) = \omega(\eta)} \qquad (C)$$

- Extension of τ to tensor products:

$$\tau(S_1 \otimes S_2) := \tau(S_1) \otimes \tau(S_2) \qquad (T)$$

S_1 and S_2 are tensors of arbitrary rank.

Definition:

Exercise:

Show that when S is a scalar function $S \in \mathfrak{F}(M)$, then:

$$\nabla_\xi S = \xi(S) = \xi^i \frac{\partial}{\partial x^i} S$$

arbitrary tensor, arb. point $\in M$, arb. tangent vector

$$\nabla_\xi S(p) := \nabla_{\dot{\gamma}} S(\gamma(t))\Big|_{t=0}$$

$$:= \frac{d}{dt}\Big|_{t=0} \tau(t,0)\big(S(\gamma(t))\big)$$

here, γ is any path through p obeying:

$$\dot{\gamma}(0) = \xi(p) \quad , \quad \gamma(0) = p$$

Absolute covariant derivative:

(for abs. derivative one is not specifying the direction.)

$$(\nabla S)\big(\underbrace{\eta_1, \dots, \eta_p, \omega_1, \dots, \omega_r, \xi}_{\text{fed to } \nabla S \text{ which is } (r,p) \text{ tensor}}\big) := \nabla_\xi S(\eta_1, \dots, \eta_p, \omega_1, \dots, \omega_r)$$

Properties of ∇:

* ∇ is a derivation: (because ∇ inherits the Leibniz rule from $\frac{d}{ds}$)

$\tau(s,t)$, $S_1(\gamma(s))$, $S_2(\gamma(s))$

$$\nabla_\xi(S_1 \otimes S_2) = \frac{d}{ds}\Big|_{s=t} \tau(S_1 \otimes S_2) = \frac{d}{ds}\Big|_{s=t} \tau(S_1) \otimes \tau(S_2)$$

$$= \left[\frac{d}{ds}\Big|_{s=t} \tau(S_1)\right] \otimes \tau(S_2)\Big|_{s=t} + \tau(S_1)\Big|_{s=t} \otimes \frac{d}{ds}\Big|_{s=t} \tau(S_2)$$

$$= (\nabla_\xi S_1) \otimes S_2 + S_1 \otimes \nabla_\xi S_2 \qquad (A)$$

* Eq. (C) implies that ∇ and contractions do commute.

Action of ∇ on tensors in a chart?

- Recall: $\boxed{\nabla_\xi \frac{\partial}{\partial x^i} = \xi^\ell \Gamma^k{}_{\ell i} \frac{\partial}{\partial x^k}}$
- Problem: Find $\nabla_\xi dx^i = ?$

Same strategy will be used below for general tensors.

- Consider $\underset{\text{tangent vector field}}{\eta} \otimes \underset{\text{cotangent vector field}}{\omega}$.
- Differentiate:

$$\nabla_\xi(\eta \otimes \omega) \overset{(A)}{=} (\nabla_\xi \eta) \otimes \omega + \eta \otimes \nabla_\xi \omega$$

- Contract: (use that ∇_ξ and contraction commute)

(by exercise above)

$$\xi(\omega(\eta)) = \nabla_\xi(\underbrace{\omega(\eta)}_{\text{scalar function}}) = \omega(\nabla_\xi \eta) + (\nabla_\xi \omega)(\eta)$$

(i.e. $\xi(\omega(\eta)) = \omega(\nabla_\xi \eta) + (\nabla_\xi \omega)(\eta)$)

$\Rightarrow$ An expression for $\nabla_\xi(\omega)(\eta)$:

$$(\nabla_\xi \omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_\xi \eta) \qquad (*)$$

Now: Choose $\omega := dx^j$ and $\eta := \frac{\partial}{\partial x^i}$

$$\Rightarrow (\nabla_\xi dx^j)\left(\frac{\partial}{\partial x^i}\right) = \underbrace{\xi\big(\overbrace{\langle dx^j, \tfrac{\partial}{\partial x^i}\rangle}^{\delta^j_i = \text{const.}}\big)}_{=0} - \langle dx^j, \nabla_\xi \tfrac{\partial}{\partial x^i}\rangle$$

Notation: $\langle \omega, \xi \rangle = \omega(\xi)$ (inner product, contraction)

$$= -\langle dx^j, \xi^\ell \Gamma^k{}_{\ell i} \frac{\partial}{\partial x^k} \rangle$$

$$= -\xi^\ell \Gamma^j{}_{\ell i}$$

$$\Rightarrow \boxed{\nabla_\xi dx^j = -\xi^\ell \Gamma^j{}_{\ell i}\, dx^i}$$

For general tensors: (by exactly same strategy as above but applied to multiple tensor products, we obtain:)

$$\nabla_\xi S(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_s)$$ (as in Eq. (*) above)

$$= \xi(S(\eta_1 \dots \eta_r, \omega_1 \dots \omega_s))$$

$$- S(\nabla_\xi \eta_1, \eta_2, \dots, \eta_r, \omega_1 \dots \omega_s) - \dots$$

$$- S(\eta_1, \dots, \nabla_\xi \eta_r, \omega_1 \dots \omega_s)$$

$$- S(\eta_1, \dots, \eta_r, \nabla_\xi \omega_1, \omega_2, \dots, \omega_s) + \dots$$

$$- S(\eta_1 \dots \eta_r, \omega_1 \dots \nabla_\xi \omega_s)$$

Choosing the basis vectors dx^i and $\frac{\partial}{\partial x^i}$ we obtain for

$$S = S^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

that $\nabla_\xi S$ reads

$$\nabla_\xi S = \xi^k S^{i_1 \dots i_p}_{j_1 \dots j_q ; k} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

with:
$$S^{i_1 \dots i_p}_{j_1 \dots j_q ; k} := S^{i_1 \dots i_p}_{j_1 \dots j_q , k} + \Gamma^{i_1}_{k\ell} S^{\ell i_2 \dots i_p}_{j_1 \dots j_q}$$
$$+ \dots + \Gamma^{i_p}_{k\ell} S^{i_1 \dots \ell}_{j_1 \dots j_q}$$
$$- \Gamma^{\ell}_{k j_1} S^{i_1 \dots i_p}_{\ell \dots j_q}$$
$$- \dots - \Gamma^{\ell}_{k j_q} S^{i_1 \dots i_p}_{j_1 \dots \ell}$$

Special cases:

- Tangent vector fields:

$$\xi^i{}_{;k} = \xi^i{}_{,k} + \xi^j \Gamma^i{}_{kj}$$

- Cotangent vector fields:

$$\omega_{i;k} = \omega_{i,k} - \omega_l \Gamma^l{}_{ki}$$

Recall: Specifying ∇ specifies parallel transport of vectors and this should specify the manifold's shape, but how?

→ Indeed, ∇ specifies Torsion & Curvature