GR for Cosmology, Achim Kempf Lecture 9

Note Title

Recall: So far, we have 2 ways to capture shape:

- Specified g $\Rightarrow$ specified distances in M
  $\Rightarrow$ implicitly specified "shape" of M

(Notice (for essay): See also my new paper 1510.02725)

Then, new:

- Specified ∇ $\Rightarrow$ specified parallel transport in M
  $\Rightarrow$ implicitly specified "shape" of M

Question:

How does ∇ determine "shape"? Through:

Torsion & Curvature!

Recall:

$$\boxed{\bar{\Gamma}^r{}_{ab} = \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij} + \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b}}$$

Notice: The antisymmetric part of Γ transforms tensorially!

$$\left.\begin{aligned}\Gamma^k_{(sym)\,ij} &:= \tfrac{1}{2}\left(\Gamma^k{}_{ij} + \Gamma^k{}_{ji}\right)\\ \Gamma^k_{(asym)\,ij} &:= \tfrac{1}{2}\left(\Gamma^k{}_{ij} - \Gamma^k{}_{ji}\right)\end{aligned}\right\}\quad \Gamma^k{}_{ij} = \Gamma^k_{(sym)\,ij} + \Gamma^k_{(asym)\,ij}$$

$$\Rightarrow \bar{\Gamma}^r_{(asym)\,ab} = \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k_{(asym)\,ij} \quad !$$

Definition: $T^k{}_{ij} := 2\,\Gamma^k_{(asym)\,ij}$ is the "Torsion tensor"

(Notice: Since Γ is not a tensor, but Γ_{asym} is, Γ_{sym} is not a tensor)

In General Relativity: one assumes torsionless ∇, i.e.: $T = 0$.

Idea: "(Extended) equivalence principle:"

Christoffel Γ will express gravitational and pseudo forces. Therefore, we require that around each $p \in M$ there exists a chart so that $\Gamma(p) = 0$ (i.e. no such forces in free fall).

This rules out the existence of torsion:

Why? The torsion is a tensor.

$\Rightarrow$ It transforms linearly with invertible Jacobian matrices

$$\bar{T}^i_{\ jk}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} T^a_{\ bc}(x)$$

$\Rightarrow$ If T_{ij} vanishes in one cds, it vanishes in all cds.

Proposition: Vice versa, if $T^i_{\ jk}(x) = 0 \ \forall x \in M$, then there is for every $p \in M$ a chart with $\Gamma^i_{\ jk}(p) = 0$.

Recall: $\underset{T^1(M)}{\xi \in}$ is autoparallel to a path $\gamma: t \to x(t)$ if

$$\boxed{\nabla_{\dot{\gamma}} \xi = 0}$$

Meaning: ξ is parallel transported along the path γ in M.

Explicitly: $\frac{d\xi^k}{dt} + \Gamma^k_{\ ij} \frac{dx^i}{dt} \xi^j = 0$

Geodesics: A curve $\gamma: t \to x(t)$ is called a geodesic if $\dot{\gamma}$ is autoparallel along γ, i.e. if

$$\boxed{\nabla_{\dot{\gamma}} \dot{\gamma} = 0}$$

Meaning: The path γ in M is such that the path's tangent vectors are parallel translates of each other.

▯ $\Rightarrow$ In charts, geodesics $x^r(t)$ obey:

$$\frac{d^2x^k}{dt^2} + \Gamma(x)^k{}_{ij}\, \frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \qquad (*)$$

▯ Theory of ordinary differential equations:

$\Rightarrow$ Given $p = \gamma(0)$, each initial condition $\xi = \dot\gamma(0)$ belongs to a unique geodesic γ_ξ of nonzero length.

subscript indicates initial condition vector

▯ <u>Notice:</u> If $\gamma_\xi(t)$ solves $(*)$ then $\gamma_\xi(\lambda t)$ also solves $(*)$ and for $\lambda \in \mathbb{R}$:

$$\gamma_{\lambda\xi}(t) = \gamma_\xi(\lambda t) \qquad (G)$$

(Exercise: verify)

"Exponential map":

▯ Consider a fixed point $p \in M$.

The exponential map is defined through:

$$\exp_p : T_p{}'(M) \to M$$

(really from a neighborhood of 0 in $T_p{}'(M)$ to a neighborhood of p in M.)

$$\exp_p : \quad \xi \to \exp_p(\xi) := \gamma_\xi(1)$$

where γ is the geodesic with $\gamma_\xi(0) = p$, $\dot\gamma_\xi(0) = \xi$.

▯ <u>Observe:</u>

From (G) we obtain:

$$\gamma_\xi(\lambda) = \gamma_{\lambda\xi}(1) = \exp_p(\lambda\xi) \qquad (E)$$

"Geodesic" or "Riemann normal" coordinates:

- $\exp_p$ is a diffeomorphism from a neighborhood of $0 \in T_p'(M) \simeq \mathbb{R}^n$ (← isomorphic) into a neighborhood of the point $p \in M$.

$\Rightarrow$ $\exp_p$ provides a chart around p:

- Choose a basis, say $e_1, e_2, \dots, e_n$ of $T_p'(M)$, then:

$$\xi = \xi^i e_i$$

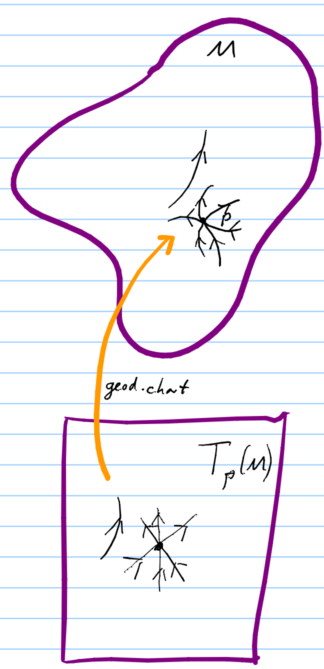
- Through $\exp_p$, the ξ^i become the coordinates of points in a neighborhood of $p \in M$:

$$(\xi^1, \dots, \xi^n) \to \exp_p(\xi^i e_i) \in M$$

- These $\{\xi^i\}$ are called "normal" or "geodesic coordinates."

$\Rightarrow$ Geodesics, γ, through p are straight lines in a normal cds about p!

- Recall (E):



$$\underbrace{\gamma_\xi(\lambda)}_{\text{for varying } \lambda \text{ one moves along the geodesic in } M.} = \exp_p(\underbrace{\lambda\xi}_{\text{for varying } \lambda \text{ one moves on a straight line in the coordinate system of the } \xi^i!})$$

- Thus: In geodesic cds, geodesics through p are straight lines of constant velocity ξ.

- Does this mean $\Gamma^k{}_{ij}(p) = 0$? No!

Geodesic eqn. at p: $\cancel{\frac{d^2x^k}{dt^2}} + \Gamma(p)^k{}_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$

Thus: $\underbrace{\left(\Gamma_{sym}{}^k{}_{ij}(p) + \Gamma_{asym}{}^k{}_{ij}(p)\right)\frac{dx^i}{dt}\frac{dx^j}{dt}}_{=0 \text{ because of type } (\text{antisymmetric})_{ij}(\text{symmetric})^{ij}} = 0$

$\Rightarrow \Gamma_{sym}{}^k{}_{ij}(p) = 0$ in geodesic cds.

$\Rightarrow$ **Indeed:** If the torsion vanishes, $\Gamma_{asym}{}^k{}_{ij}(p) = \frac{1}{2} T^k{}_{ij}(p) = 0$ then for each $p \in M$ there exists a chart in which the entire gravity and pseudo force field vanishes at p:

$$\Gamma^k{}_{ij}(p) = 0$$

Note:
Quantum fluctuations may induce torsion!
So, let's nevertheless ask:

What would torsion mean, geometrically?

Abstract definition of Torsion:

- Assume ξ_1 and ξ_2 are tangent vectors at $p \in M$:

Then, the **Torsion map** is defined as:

$$\mathcal{T}: T_p^1(M) \times T_p^1(M) \rightarrow T_p^1(M)$$

$$\mathcal{T}: \xi_1, \xi_2 \longrightarrow \mathcal{T}(\xi_1, \xi_2) := \overbrace{\nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2]}^{\text{This will be the amount by which an infinitesimal parallelogram spanned by } \xi_1 \text{ and } \xi_2 \text{ does not close.}}$$

- It is used to define the **Torsion tensor**, T, (for proof it's a tensor, see Straumann)

$$T \in T_p{}^1{}_2(M)$$

through:

feeding 1 covector & 2 vectors to a (1,2) tensor yields a number $\rightarrow$

$$T(\omega, \xi_1, \xi_2) := \overbrace{\langle \underbrace{\omega}_{\in T_{p,1}(M)}, \underbrace{\mathcal{T}(\xi_1, \xi_2)}_{\in T_p{}^1(M)} \rangle}^{\text{contraction yields a number}} \in \mathbb{R}$$

we could also write: $= \omega(\mathcal{T}(\xi_1, \xi_2))$

Compare with prior definition:

- Choose canonical bases $\omega := dx^k$, $\xi_1 := \frac{\partial}{\partial x^i}$, $\xi_2 := \frac{\partial}{\partial x^j}$:

- $$T^k{}_{ij} := dx^k\left(T\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\right)$$

$$= \left\langle dx^k, T\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\right\rangle \quad \text{(more convenient notation)}$$

$$= \left\langle dx^k, \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} - \underbrace{\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]}_{\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}-\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right)f = 0 \ \forall f}\right\rangle$$

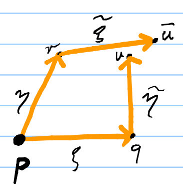
Recall: $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k{}_{ij}\frac{\partial}{\partial x^k}$

$$= \left\langle dx^k, \Gamma^r{}_{ij}\frac{\partial}{\partial x^r} - \Gamma^r{}_{ji}\frac{\partial}{\partial x^r}\right\rangle = \Gamma^r{}_{ij}\delta^k_r - \Gamma^r{}_{ji}\delta^k_r$$

- $\Rightarrow$ $$\boxed{T^k{}_{ij} = \Gamma^k{}_{ij} - \Gamma^k{}_{ji}}$$

Geometric meaning of torsion? Parallelograms would not close!

Travel from p infinitesimally in ξ and then η direction, and compare with the reverse. (In flat space: $x^r+\eta^r+\xi^r = x^r+\xi^r+\eta^r$.)



$\xi, \eta \in T_p'$

$\tilde\xi \in T_r'$

$\tilde\eta \in T_q'$

Recall parallel transport: $\nabla_{\dot\gamma} v = 0$

$$\frac{dv^k}{dt} + \Gamma^k{}_{ij}\frac{dx^i}{dt}v^j = 0$$

$\tilde\xi(r) = ?$

$$\tilde\xi^k(x^i+\eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i)$$

Now use $v := \xi$, $\frac{dx^i}{dt} = \eta^i$:

$$= \xi^k(x^i) - \Gamma(x)^k{}_{ij}\eta^i\xi^j$$

$\Rightarrow$ Cds. of $\bar u$: $x^a+\eta^a+\xi^a-\Gamma(x)^a{}_{ij}\eta^i\xi^j$

Analogously obtain: Cds. of u: $x^a + \xi^a + \eta^a - \Gamma(x)^a{}_{ij}\,\xi^i\eta^j$

$\Rightarrow$ Cd. distance from u to $\bar{u}$ is: $\left(\Gamma(x)^a{}_{ji} - \Gamma(x)^a{}_{ij}\right)\eta^i\xi^j = \underset{\text{Torsion!}}{T^a_{ji}}\,\eta^i\xi^j$ ✓

Comment: We had:

$$\tilde{\xi}^k(x^i + \eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i) = \xi^k(x^i) - \Gamma(x)^k{}_{ij}\,\eta^i\xi^j$$

this is also: $= \xi^k(x^i) - \left(\eta^i\xi^k{}_{,i} + \Gamma(x)^k{}_{ij}\,\eta^i\xi^j\right) + \eta^i\xi^k{}_{,i}$

$$= \xi^k(x^i) - \eta^i\xi^k{}_{;i} + \eta^i\xi^k{}_{,i}$$

Thus: cd distance from u to $\bar{u}$ is:

$$\left(\cancel{x^a} + \cancel{\eta^a} + \cancel{\xi^a} - \eta^i\xi^k{}_{;i} + \eta^i\xi^k{}_{,i}\right) - \left(\cancel{x^a} - \cancel{\xi^a} - \cancel{\eta^a} + \xi^i\eta^k{}_{;i} - \eta^i\xi^k{}_{,i}\right) = T^a_{ji}\,\eta^i\xi^j$$

Recall that indeed: $T: \eta, \xi \to T(\eta, \xi) = \nabla_\eta \xi - \nabla_\xi \eta - [\eta, \xi]$

Curvature:

Assume ξ_1, ξ_2 and ξ_3 are tangent vectors at $p \in M$.

- The curvature map, R, is defined through:

$$R: \xi_1, \xi_2, \xi_3 \to \underbrace{R(\xi_1, \xi_2)}_{\text{an operator, or map, acting on } \xi_3}\xi_3 = \overbrace{\left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1, \xi_2]}\right)\xi_3}^{\in T_p{}^1(M)}$$

- It defines the curvature tensor, R,

can be fed one covector and 3 vectors to yield a number

$$R \in T^1_3(M)$$

through:

$$R(\omega, \xi_1, \xi_2, \xi_3) := \langle \omega, \overbrace{R(\xi_1, \xi_2)\xi_3 \rangle}^{= \omega(R(\xi_1,\xi_2)\xi_3)} \in \mathbb{R}$$

In a chart:

$$R^i{}_{jkl} = \left\langle dx^i, R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j}\right\rangle$$

$$= \left\langle dx^i, \left(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^l}} - \nabla_{\frac{\partial}{\partial x^l}} \nabla_{\frac{\partial}{\partial x^k}} - \nabla_{\underbrace{[\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}]}_{=0}}\right)\frac{\partial}{\partial x^j}\right\rangle$$

$$= \left\langle dx^i, \nabla_{\frac{\partial}{\partial x^k}} \Gamma^s{}_{lj} \frac{\partial}{\partial x^s} - \nabla_{\frac{\partial}{\partial x^l}} \Gamma^s{}_{kj} \frac{\partial}{\partial x^s} \right\rangle$$

$$= \left\langle dx^i, \left(\Gamma^s{}_{lj,k} + \Gamma^r{}_{lj}\Gamma^s{}_{kr} - \Gamma^s{}_{kj,l} - \Gamma^r{}_{kj}\Gamma^s{}_{lr}\right)\frac{\partial}{\partial x^s}\right\rangle$$

$$= \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \underbrace{\Gamma^s{}_{lj}\Gamma^i{}_{ks} - \Gamma^s{}_{kj}\Gamma^i{}_{ls}}_{\text{(at origin of geodesic cds they vanish.)}}$$

Curvature tensor's meaning?

Intuition:

- Contains derivatives of Γ $\Rightarrow$
- expresses variation in gravitational forces
  $\Rightarrow$
- expresses the strength and direction of "tidal forces".

Geometry:

- Curvature expresses noncommutativity of two parallel transports, namely:

Proposition: (Ricci Identity)

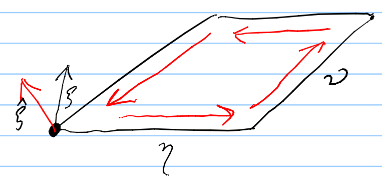
Assume the torsion vanishes and that ξ is a vector field. Then:

$$\boxed{\xi^a{}_{;cd} - \xi^a{}_{;dc} = R^a{}_{cdb}\,\xi^b}$$

(here: $\xi^a{}_{;cd} := \xi^a{}_{;c;d}$ etc.)

Remark: (a bit messy to derive because need Taylor expansion, see, e.g., text by Stewart or Einstein)

It implies that for parallel transport along infinitesimal parallelogram:



$$\boxed{(\hat{\xi} - \xi)^a \approx \eta^b v^c R^a{}_{bcd}\,\xi^d}$$

Proof of Ricci identity:

- Assume ξ, η, v are vector fields.
- Then, $R(\xi,\eta)v := \nabla_\xi(\nabla_\eta v) - \nabla_\eta(\nabla_\xi v) - \nabla_{[\xi,\eta]} v$ reads

use: $\nabla_\eta v = \nabla_{\eta^i \frac{\partial}{\partial x^i}}\left(v^j \frac{\partial}{\partial x^j}\right) = \eta^i \nabla_{\frac{\partial}{\partial x^i}}\left(v^j \frac{\partial}{\partial x^j}\right) = \eta^i v^j{}_{;i} \frac{\partial}{\partial x^j} \ldots$

in basis:
$$R^a{}_{bcd}\,\xi^b \eta^c v^d = \left(v^a{}_{;d}\,\eta^d\right)_{;c}\xi^c - \left(v^a{}_{;d}\,\xi^d\right)_{;c}\eta^c$$
$$- v^a{}_{;d}\underbrace{\left(\eta^d{}_{;c}\,\xi^c - \xi^d{}_{;c}\,\eta^c\right)}$$

used Torsion $= T(\xi_1,\xi_2) := \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1,\xi_2] = 0$

i.e.: $[\xi,\eta] = \nabla_\xi \eta - \nabla_\eta \xi$

Terms cancel:

$$\Longrightarrow R^a{}_{bcd}\,\xi^b \eta^c v^d = \left(v^a{}_{;d;c} - v^a{}_{;c;d}\right)\xi^c \eta^d$$

- True $\forall \xi, \eta$ $\Longrightarrow R^a{}_{bcd}\, v^d = v^a{}_{;cb} - v^a{}_{;bc}$ ✓

The "Bianchi Identities":

- They are automatic relations among torsion and curvature, by construction.

- Preparation: $\boxed{\nabla \text{ for maps!}}$

Consider an arbitrary $\mathcal{F}(M)$-linear map:

(e.g. Torsion or Curvature map)

$$K: \underbrace{\xi_1 \times \xi_2 \times \ldots \times \xi_r}_{\text{tangent vectors}} \longrightarrow \underbrace{K(\xi_1, \ldots, \xi_r)}_{\text{tangent vector}}$$

i.e. at each $p \in M$:

$$K: T_p(M)^r \longrightarrow T_p(M)^1$$

- We can view K as a tensor $\tilde{K} \in T_p(M)^1_r$,

(as we did for R and T)

namely:

$$\tilde{K}(\omega, \xi_1, \ldots, \xi_r) := \langle \omega, K(\xi_1, \ldots, \xi_r) \rangle$$

- Now let the usual derivative of the tensor $\tilde{K}$ define the derivative of the map K:

$$\langle \omega, (\underbrace{\nabla_\xi}_{\substack{\text{new concept:}\\ \text{covariant derivative}\\ \text{of a map } K: T_p(M)^r \to T_p(M)^1}} K)(\xi_1, \ldots, \xi_r) \rangle := \underbrace{\nabla_\xi \tilde{K}(\omega, \xi_1, \ldots \xi_r)}_{\substack{\text{usual cov. derivative}\\ \text{of a } (1,r) \text{ tensor}\\ \text{when fed one covector \& } r \text{ vectors}}}$$

Using ∇ for map:

1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi,\eta)\nu = \sum_{\text{cyclic}}\Big(T(T(\xi,\eta),\nu) + (\nabla_\xi T)(\eta,\nu)\Big)$$

2nd Bianchi Identity:

$$\sum_{\text{cyclic}}\Big((\nabla_\xi R)(\eta,\nu) + R(T(\xi,\eta),\nu)\Big) = 0$$

with obvious simplification in case $T = 0$.

Note: They are automatically obeyed equations, just like (or like the homogeneous Maxwell equations) any set of lin. operators obeys the Jacobi identity with respect to $[\ ,\ ]$. Indeed that's why:

Proof of 1st Bianchi: (assuming no torsion)

$$\sum_{\text{cyclic}} R(\xi,\eta)\nu = 0$$

Indeed:

$$\underbrace{\sum_{\text{cyclic}} R(\xi,\eta)\nu}_{\parallel} = (\nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi)\nu - \nabla_{[\xi,\eta]}\nu + \text{cyclic}$$

shift by 1 cyclically ↓ shift by 1 cyclically ↓

$$= \nabla_\xi(\nabla_\eta\nu - \nabla_\nu\eta) - \nabla_{[\eta,\nu]}\xi + \text{cyclic}$$

Exercise: Prove that (without torsion): $\boxed{\nabla_\eta\nu - \nabla_\nu\eta = [\eta,\nu]}$ (easy!)

$$= \underbrace{\nabla_\xi[\eta,\nu] - \nabla_{[\eta,\nu]}\xi}_{\parallel \text{ because again } \nabla_a b - \nabla_b a = [a,b]} + \text{cyclic}$$

$$= [\xi,[\eta,\nu]] + \text{cyclic}$$

$= 0$ by Jacobi identity for all lin. maps.

Recall:

Assume A, B, C are linear maps $V \to V$

Then: $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$

i.e., the Jacobi identity holds.

Proof: Simply spell out the commutators.

Remark: This means that, e.g., in quantum mechanics, all objects that that need to be representable as operators on the Hilbert space must obey the Jacobi identity, e.g., generators of symmetries.

This is why the Jacobi identity is one of the axioms of Lie Algebras.