QFT for Cosmology, Achim Kempf, Lecture 10

Note Title

Recall:

(because the Lagrangian framework treats space and time in the same way)

* Hamiltonian formulations are suitable for quantization.
* Lagrangian formulations are suitable to achieve general relativistic covariance.

→ **Strategy:**

SR, 1^{st}Q Hamiltonian formalism	step 1: Legendre transform (equivalence) →	SR, 1^{st}Q. Lagrangian formalism
		↓ step 2: allow curvature
GR, 1^{st}Q Hamiltonian formalism	← step 3: Legendre transform (equivalence)	GR, 1^{st}Q Lagrangian formalism
↓ step 4		↓
GR, 2^{nd}Q Hamiltonian formalism	← Dyson Schwinger eqns are same (equivalence) →	GR, 2^{nd}Q Lagrangian formalism (Path integral of QFT)

We already started step 1:

$$H[\phi,\pi,t] \quad \underset{\pi(x,t):=\frac{\delta L}{\delta \beta(x,t)} \quad (T^{-1})}{\overset{\beta(x,t):=\frac{\delta H}{\delta \pi(x,t)} \quad (T)}{\rightleftarrows}} \quad L[\phi,\beta,t]$$

Proposition: These equations of motion are equivalent:

Hamiltonian eqns. of motion:	Lagrangian eqns. of motion:
$\dot\phi(x,t) = \dfrac{\delta H[\phi,\pi,t]}{\delta \pi(x,t)}$ (H1)	$\dot\phi(x,t) = \beta(x,t)$ (L1)
$\dot\pi(x,t) = -\dfrac{\delta H[\phi,\pi,t]}{\delta \phi(x,t)}$ (H2)	$\dfrac{\delta L}{\delta\phi(x,t)} = \dfrac{d}{dt}\dfrac{\delta L}{\delta \beta(x,t)}$ (L2)

Proof: We need to show that $(H1 \wedge H2) \overset{T}{\Longleftrightarrow} (L1 \wedge L2)$.

The case "$\Longrightarrow$"

- **Show L1:** $\dot\phi \overset{(H1)}{=} \frac{\delta H}{\delta \pi} \overset{(T)}{=} \beta$ ✓
- **Show L2:**

$$\frac{d}{dt}\frac{\delta L(\phi,\beta,t)}{\delta \beta} \overset{(T^{-1})}{=} \frac{d}{dt}\pi$$

$$\overset{(H2)}{=} -\frac{\delta H(\phi,\pi,t)}{\delta \phi}$$

$$\overset{\text{by def. of } L}{=} -\frac{\delta}{\delta\phi}\left(\int \beta(\phi,\pi)\pi\, d^3x - L(\phi,\beta(\phi,\pi),t)\right)$$

$$= -\cancel{\frac{\delta\beta}{\delta\phi}\pi} + \frac{\delta L}{\delta\phi} + \cancel{\underbrace{\frac{\delta L}{\delta\beta}}_{\pi(T)}\frac{\delta\beta}{\delta\phi}}$$ ✓

Exercise: The case "$\Longleftarrow$".

Result so far:

- Legendre transform to Lagrangian formulation

$\Longrightarrow$ Eqns of motion can be cast in the form L1, L2, i.e.:

(Notice: Only a time derivative, no occurrence of space derivatives?) $\rightarrow$

$$\frac{\delta L}{\delta\phi(x,t)} = \frac{d}{dt}\frac{\delta L}{\delta\beta(x,t)}, \qquad \beta(x,t) = \dot\phi(x,t)$$

But: How is that advantageous? These equations still seem to treat time differently than space!

Analysis of L1, L2:

We notice: * The term $\frac{\delta L}{\delta \phi(x,t)}$ is the total derivative with respect to all occurrences of ϕ in L, including occurrences of $\frac{\partial}{\partial x_i}\phi(x,t)$ in L.

* Why? Because of the definition of $\frac{\delta}{\delta \phi}$:

$$\frac{\delta L}{\delta \phi(x,t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(L\left[\{\phi(x',t)+\varepsilon\,\delta^3(x'-x)\}_{x'\in\mathbb{R}^3}\right] - L\left[\{\phi(x',t)\}_{x\in\mathbb{R}^3}\right]\right)$$

E.g.: $F[u] := \int \sin(x)\left(\frac{d}{dx}u(x)\right)dx$ Is $\frac{\delta F}{\delta u(x)} = 0$? No:

$$= -\int \cos(x)\, u(x)\, dx \quad \text{(We assume } u(x)\to 0 \text{ at boundaries)}$$

$$\Rightarrow \frac{\delta F}{u(x)} = -\cos(x)$$

$\Longrightarrow$ L1, L2 will contain nontrivial time and space derivatives.

* Is there a systematic way to evaluate the derivatives with respect to $\frac{\partial \phi}{\partial x_i}$?

Lemma: Consider any functional Z of the form:

$$Z[f] = \int \text{polynomial}\left(\frac{d}{dx}f\right) dx$$

Then: $$\frac{\delta Z}{\delta f(x)} = -\frac{d}{dx}\,\frac{\delta Z}{\delta\left(\frac{d}{dx}f\right)}$$

On the right hand side we view $\frac{d}{dx}f$ as an independent function.

Example:

Notation: $\partial_x f(x) = \frac{d}{dx} f(x)$

$$Z[f] := \int_{\mathbb{R}} \left(\partial_{x'} f(x')\right)^2 dx'$$

- If we view $\partial_x f$ as an independent function, then we obtain of course:

$$\frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = 2\, \partial_x f(x)$$

- Our lemma claims, therefore:

$$\frac{\delta Z[f]}{\delta f(x)} = -\partial_x \frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = -2\, \partial_x \partial_x f(x)$$

- Let us verify this from first principles!

Indeed:

$$\frac{\delta}{\delta f(x)} Z[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\int_{\mathbb{R}} \left(\partial_{x'}\left(f(x') + \varepsilon\,\delta(x-x')\right)\right)^2 dx' - \int_{\mathbb{R}} \left(\partial_{x'} f(x')\right)^2 dx'\right]$$

$$\overset{\lim \varepsilon \to 0}{=} 2 \int_{\mathbb{R}} \left(\partial_{x'} f(x')\right)\left(\partial_{x'} \delta(x-x')\right) dx'$$

$$\underset{\text{parts}}{\overset{\text{int. by}}{=}} -2 \int_{\mathbb{R}} \left(\partial_{x'}^2 f(x')\right) \delta(x-x')\, dx' + \cancel{\text{boundary term}}$$

$$= -2\, \partial_x^2 f(x) \quad \checkmark$$

Recall L2: $$\frac{\delta L[\phi,\beta,t]}{\delta\phi(x,t)} = \frac{d}{dt}\frac{\delta L[\phi,\beta,t]}{\delta\beta(x,t)}$$

Use lemma:

$$\frac{\delta L[\phi,\beta,t]}{\delta\phi(x,t)} = \frac{\delta L[\phi,\partial_1\phi,\partial_2\phi,\partial_3\phi,\beta,t]}{\delta\phi(x,t)} - \sum_{j=1}^{3}\frac{\partial}{\partial x^j}\frac{\delta L[\phi,\partial_1\phi,\partial_2\phi,\partial_3\phi,\beta,t]}{\delta\left(\partial_j\phi(x,t)\right)}$$

$\Rightarrow$ L2 takes the form:

$$\frac{\delta L[\phi,\partial_j\phi,\beta,t]}{\delta\phi(x,t)} - \sum_{j=1}^{3}\frac{\partial}{\partial x^j}\frac{\delta L[\phi,\partial_j\phi,\beta,t]}{\delta\left(\partial_j\phi(x,t)\right)} = \frac{d}{dt}\frac{\delta L[\phi,\partial_j\phi,\beta,t]}{\delta\beta(x,t)}$$

Recall also L1: $$\beta(x,t) = \dot\phi(x,t)$$

$\rightsquigarrow$ One is tempted to write:

$$\frac{\delta L[\phi,\partial_j\phi,t]}{\delta\phi(x,t)} \overset{?}{=} \sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\frac{\delta L[\phi,\partial_\mu\phi,t]}{\delta\left(\partial_\mu\phi(x,t)\right)} \qquad \text{with: } \partial_0 := \frac{d}{dt}$$

However: Here, we must remember that here the true variable is β, and that we can set $\beta = \dot\phi$ only after functional differentiation.

Ramification? Can we use the lemma to write

$$\frac{\delta L[\phi,t]}{\delta \phi(x,t)} = 0$$

for the Euler Lagrange field equations? **No!**

Because: to apply the lemma to the derivative $\frac{\partial}{\partial t}\phi$, one would need that L possesses a t-integration:

(**Lemma:** For any functional Z of the form:

$$Z[f] = \int \text{polynomial}\left(\frac{d}{dx} f\right) dx$$

we have: $\frac{\delta Z}{\delta f(x)} = -\frac{d}{dx}\frac{\delta Z}{\delta\left(\frac{d}{dx}f\right)}$)

→ **The "Action functional":**

Definition: $S[\phi] := \int_{\mathbb{R}} L[\phi,t]\,dt$

$S[\phi]$ is called the "action of the field evolution $\phi(x,t)$"

Then, the "Euler Lagrange field equations" are

$$\boxed{\frac{\delta S[\phi,\partial_\mu\phi]}{\delta\phi(x,t)} - \sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\frac{\delta S[\phi,\partial_\mu\phi]}{\delta(\partial_\mu\phi)} = 0}$$

or equivalently:

$$\boxed{\frac{\delta S[\phi]}{\delta\phi(x,t)} = 0}$$

"The action principle"

▫ Notice that the action principle, spelled out, reads:

$$0 = \frac{\delta S[\phi]}{\delta \phi(x,t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(S[\{\phi(x') + \varepsilon\, \delta^4(x-x')\}_{x' \in \mathbb{R}^4}] - S[\{\phi(x')\}_{x \in \mathbb{R}^4}]\Big)$$

Example:

The Klein Gordon action:

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} (\partial_0 \phi)^2 - \sum_{j=1}^{3} (\partial_j \phi)^2 - m^2 \phi^2 \; d^4x$$

▫ Using either the action principle or directly the Euler Lagrange field equations, one obtains the Klein Gordon equation (Exercise: verify):

$$\partial_0^2 \phi - \Delta \phi + m^2 \phi = 0, \text{ i.e., } (\Box + m^2)\phi(x,t) = 0$$

▫ **Definitions:**

* The action's integrand is called the "**Lagrange density**" $\mathcal{L}(x,t)$:

$$S[\phi] = \int_{\mathbb{R}^4} \mathcal{L}(x,t)\; d^4x$$

* One often formally writes:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \qquad (\mathcal{L})$$

* Notation often used in General Relativity:

a.) $\phi_{,\mu}(x,t) := \frac{\partial}{\partial x^\mu}\phi(x,t)$

b.) Twice occuring indices are to be summed over (Einstein summation convention):

E.g., equation (ℒ) can be written as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

c.) One defines the **metric tensor** $g_{\mu\nu}(x,t)$. More about it soon. In special relativity in inertial rectangular coordinate system, we have:

$$g_{\mu\nu}(x,t) = \eta_{\mu\nu} := \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

- Using these definitions, the K.G. action now reads:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 \, d^4x$$

↑ the inverse matrix to $g_{\mu\nu}$. In special relativity, both are the same: $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

- The E.L. eqns read

$$\frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta \phi(x,t)} = \partial_\mu \frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta(\phi_{,\mu}(x,t))}$$

and yield
$$-m^2\phi = \partial_\mu g^{\mu\nu}\phi_{,\nu}$$

i.e., of course: $(\Box + m^2)\phi = 0$

We have now completed Step 1:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{c}\text{SR, } 1^{st}\text{Q} \\ \text{Hamiltonian} \\ \text{formalism}\end{array} \arrow[r, "\begin{array}{c}\text{step 1} \\ \text{Legendre transform}\end{array}", "\text{(equivalence)}"'] & \begin{array}{c}\text{SR, } 1^{st}\text{Q.} \\ \text{Lagrangian} \\ \text{formalism}\end{array} \arrow[d, "\begin{array}{c}\text{allow} \\ \text{curvature}\end{array}", "\text{step 2}"'] \\
\begin{array}{c}\text{GR, } 1^{st}\text{Q} \\ \text{Hamiltonian} \\ \text{formalism}\end{array} \arrow[d, "\text{step 4}"] & \begin{array}{c}\text{GR, } 1^{st}\text{Q} \\ \text{Lagrangian} \\ \text{formalism}\end{array} \arrow[l, "\begin{array}{c}\text{step 3} \\ \text{Legendre transform}\end{array}"', "\text{(equivalence)}"] \arrow[d, dashed] \\
\begin{array}{c}\text{GR, } 2^{nd}\text{Q} \\ \text{Hamiltonian} \\ \text{formalism}\end{array} \arrow[r, leftrightarrow, "\text{Dyson Schwinger eqns are same}", "\text{(equivalence)}"'] & \begin{array}{c}\text{GR, } 2^{nd}\text{Q} \\ \text{Lagrangian formalism} \\ \text{(Path integral of QFT)}\end{array}
\end{tikzcd}$$

Now that we have a beautifully covariant Lagrangian formulation:

Step 2: How to allow for curvature of space-time?

Strategy:

A. Within special relativity, allow not just inertial rectangular coordinate systems but allow arbitrary coordinate systems.

B. Allow arbitrary coordinate systems and allow curvature.

A. Arbitrary coordinate systems

- Reconsider the K.G. action:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 \, d^4x$$

- If we change to arbitrary coordinates

$$x^\mu \rightarrow \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then:

$$\phi(x) \rightarrow \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$$

(recall that $\sum_{\nu=0}^{3}$ is implied)

$$\frac{\partial}{\partial x^\mu}\phi(x) \rightarrow \frac{\partial}{\partial \tilde{x}^\mu}\tilde{\phi}(\tilde{x}) = \left(\frac{\partial}{\partial x^\nu}\phi(x(\tilde{x}))\right)\frac{\partial x^\nu}{\partial \tilde{x}^\mu}$$

- Therefore, if we transform

$$g^{\mu\nu}(x) \rightarrow \tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x(\tilde{x}))$$

then we have that this term in the action

$$g^{\mu\nu}(x)\,\phi_{,\mu}(x)\,\phi_{,\nu}(x)$$

is numerically the same in all coordinate systems:

$$g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi(x)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x)\right) \rightarrow \tilde{g}^{\mu\nu}(\tilde{x})\left(\frac{\partial}{\partial \tilde{x}^\mu}\tilde{\phi}(\tilde{x})\right)\left(\frac{\partial}{\partial \tilde{x}^\nu}\tilde{\phi}(\tilde{x})\right)$$

$$= g^{\alpha\beta}(x)\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\frac{\partial x^\delta}{\partial \tilde{x}^\mu}\frac{\partial x^\epsilon}{\partial \tilde{x}^\nu}\left(\frac{\partial}{\partial x^\delta}\phi(x)\right)\left(\frac{\partial}{\partial x^\epsilon}\phi(x)\right)$$

$$= g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi(x)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x)\right) \text{ because } \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\mu} = \delta^\delta_\alpha$$

✓

Terminology:

* We say that we let $g^{\mu\nu}(x)$ transform as a contravariant tensor of rank 2.

  because upper indices — because 2 indices

* With $g^{\mu\nu}(x)\, g_{\nu\rho}(x) = \delta^\mu{}_\rho$ we have

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x(\tilde{x}))$$

which is called a covariant rank 2 tensor.

Is $S[\phi]$ now coordinate system independent?

No, not yet!

Recall:

As $x^\mu \to \tilde{x}^\mu(x)$ the integral measure changes by a Jacobian factor:

$$\int f(x)\, d^4x \longrightarrow \int \underbrace{\tilde{f}(\tilde{x})}_{= f(\tilde{x})} \underbrace{\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right)}_{\text{a coordinate-dependent term!}} d^4\tilde{x}$$

A compensating term is needed:

How can we modify the action $S[\phi]$ so that:

* there is no modification in cartesian coordinates
* the modification compensates the Jacobian term.

Solution:

Modify the action to include a "Volume factor":

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)\overbrace{\sqrt{-\det(g_{\mu\nu})}}\, d^4x$$

The volume factor:

* When $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $\sqrt{-\det(g)} = 1$ ✓

* Lemma: When $x^\mu \to \tilde{x}^\mu(x)$ then:

short for $\sqrt{-\det(g_{\mu\nu})}$

$$\sqrt{|g|} \longrightarrow \sqrt{|\tilde{g}|} = \det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right)\sqrt{|g|}$$

Therefore, we have now in special relativity that the action $S[\phi]$ of a field ϕ comes out the same number, independently of one's choice of coordinate system:

$$S[\phi] \longrightarrow \tilde{S}[\tilde{\phi}] = \int \tilde{\mathcal{L}}\sqrt{|\tilde{g}|}\, d^4\tilde{x}$$

$$= \int \mathcal{L}\det\left(\frac{\partial \tilde{x}}{\partial x}\right)\det\left(\frac{\partial x}{\partial \tilde{x}}\right)\sqrt{|g|}\, d^4x$$

$$= \int \mathcal{L}\det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \tilde{x}^\nu}\right)\sqrt{|g|}\, d^4x$$

$$= \int \mathcal{L}\det\left(\delta^\mu{}_\nu\right)\sqrt{|g|}\, d^4x = \int \mathcal{L}\sqrt{|g|}\, d^4x$$

$$= S[\phi]$$

B. How to allow curvature?

* The trivial metric $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ can look very nontrivial in generic coordinate systems: $g_{\mu\nu}(x) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$

* But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

→ These metrics belong to spaces with curvature. We need not change the action's formula: Just allow arbitrary metrics $g_{\mu\nu}(x)$.

□ We saw that in generic (i.e. arbitrarily chosen) coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x)$, the metric tensor $\tilde{g}_{\mu\nu}(\tilde{x})$ is given by:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta} \qquad (c)$$

⇒ In special relativity, in arbitrary coordinates, the metric $g_{\mu\nu}$ is a position-dependent matrix of the form (c).

* We notice that $g_{\mu\nu}(x)$ is always symmetric $g_{\mu\nu}(x) = g_{\nu\mu}(x)$

Key Question:

Can any arbitrary function obeying $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ arise from

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

by changing coordinates according to $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta}$?

Answer: No! The others describe "curved" spacetimes.

A given spacetime can be described by any one of an equivalence class $[g]$ of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

Definition: Each equivalence class $[g]$ is called a Riemannian or Lorentzian structure, depending on the signature of the metric.

How many Lorentzian or Riemannian structures are there?

Q: How many independent degrees of freedom D (i.e. independent functions) describe a spacetime fully?

A: In n dimensions, the metric g has n^2 component functions $g_{\mu\nu}(x)$.

Because of $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, only $n(n+1)/2$ are independent.

But we can choose n functions $\tilde{x}^\mu(x)$ in $\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} g_{\alpha\beta}$.

A: $D = \underbrace{n(n+1)/2}_{\text{\# of indep elements of a symmetric } n\times n \text{ matrix } g_{\mu\nu}} - \overbrace{n}^{\text{\# of change of coordinate functions } y^\mu = y^\mu(x)}$

Examples: For $n = 1+3$, have $D = 6$. For $n = 2$, have $D = 1$.