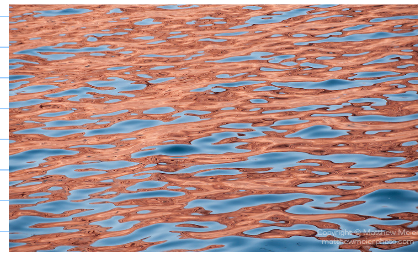
QFT for Cosmology, Achim Kempf, Lecture 4

Note Title

From Heisenberg to Schrödinger picture

Water:

$\phi(x,t)$



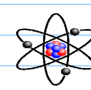
Probe amplitudes, e.g., with a cork:

Quantum field:

$\hat{\phi}(x,t)$

How to visualize an operator-valued field ?

Probe amplitudes, e.g., with atoms (lecture 8):



For now...

Assume we have some means to measure

$$\hat{\phi}(x,t)$$

at a time t for all $x \in \mathbb{R}^3$.

Q: Why possible in principle?

A: Because $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$ and $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \quad \forall x, x' \in \mathbb{R}^3$

Note: The $\hat{\phi}(x,t) \; \forall x \in \mathbb{R}^3$ are a maximal set of commuting observables.

⇒ At each x obtain real-valued measurement outcome, $\phi(x)$.

Analogous to measuring $\hat{q}_a$ and obtaining measurement outcomes q_a.

Definition: Assume that $\phi: \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary function. Then we define

$$|\phi\rangle \in \mathcal{H}$$

to be the joint eigenvector of all $\hat{\phi}(x,t)$ obeying

unique up to a phase ↗ ↖ i.e. for all $x \in \mathbb{R}^3$

$$\hat{\phi}(x,t)\,|\phi\rangle = \phi(x)\,|\phi\rangle \qquad \text{for all } x \in \mathbb{R}^3$$

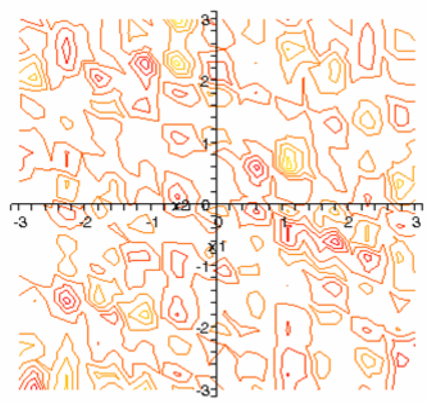
\# of particles

Analogous to: $\hat{q}_a(t)\,|q\rangle = q_a\,|q\rangle$ for all $a = 1, \dots, 3N$

Example: Assume system is in the vacuum state $|\Omega\rangle$.

What is then a typical measurement outcome $\phi(x)$?

Shown are the level curves of a typical measurement outcome $\phi(x)$.



The measurement collapses the system into the new state:

$$|\phi\rangle \in \mathcal{H}$$

Analogous to a state $|q\rangle = |q_1, q_2, q_3, \dots\rangle$

$|\phi\rangle$ is a joint eigenstate of all $\hat{\phi}(x,t)$: $\hat{\phi}(x,t)\,|\phi\rangle = \phi(x)\,|\phi\rangle$

How to calculate in the Schrödinger picture?

Preparations:

Hilbert basis: The set $\{|\phi\rangle\}$ of all joint eigenvectors of the $\hat{\phi}(x,t)$ for all $x \in \mathbb{R}^3$ can be used to form a "complete ON basis" of $\mathcal{H}$. (up to functional analytic subtleties).

Resolution of the identity:

$\Rightarrow$ For any $|\Psi\rangle \in \mathcal{H}$ we have:

$$|\Psi\rangle = \int_{\phi \in L^2(\mathbb{R}^3)} |\phi\rangle\langle\phi|\Psi\rangle \, D[\phi]$$

← it's more subtle really

analogous to: $|\psi\rangle = \int |q\rangle \underbrace{\langle q|\psi\rangle}_{\text{"}} d^3q$

Wave function: $\psi(q)$

The "Wave functional"

Recall QM:

- Assume $\{\hat{q}_i\}_{i=1}^{N}$ is compl. set of commuting observables, with joint eigenvectors $|q\rangle$ obeying: $\hat{q}_i|q\rangle = q_i|q\rangle$.

- Then the function ψ, given by $\psi(q) = \langle q|\psi\rangle$ is called the "wave function" of $|\psi\rangle$ in the $\{\hat{q}_i\}$ basis.

Example: $\{\hat{p}_i\}$ yield mom. wave functions $\psi(p) = \langle p|\psi\rangle$

$p = \{p_1, p_2, \ldots, p_N\}$

In QFT:

or, e.g., also the $\{\hat{\pi}(x)\}$.

E.g., $\{\hat{\phi}(x)\}_{x \in \mathbb{R}^3}$ is compl. set of com. observables

with joint eigenvectors $|\phi\rangle$ obeying $\hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$.

- Then, Ψ, given by

$\{|\phi\rangle\}$ form field ON eigen basis

(Convention: square bracket because argument is a function)

$$\Psi[\phi] := \langle \phi | \Psi \rangle$$

(called a "functional" because argument is a function)

[alternatively could use e.g. joint eigenbasis of the $\hat{\pi}(x,t)$.

is called the "wave functional".

Interpretation of $\Psi[\phi]$?

e.g., vacuum $|\Psi_0\rangle$

- Assume the system is in an arbitrary state $|\Psi\rangle \in \mathcal{H}$ at t.

- If measuring now $\hat{\phi}(x,t)$ at all $x \in \mathbb{R}^3$ what is the probability amplitude for finding, say, the values $\phi(x)$?

Answer: $prob[|\Psi\rangle \to |\phi\rangle] = |\langle \phi | \Psi \rangle|^2 = |\Psi[\phi]|^2$

Q: The eqn. of motion for $\Psi[\phi,t]$?

A: The QFT Schrödinger equation!

- For <u>every</u> quantum theory, we have in the Schrödinger picture of the time evolution:

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

- Which form does it take for $\Psi[\phi,t]$?

- Here in QFT:

now independent of time!

$$\hat{H} = \int \frac{1}{2}\left(\hat{\pi}^2(x) + \hat{\phi}(x)\left(-\Delta + m^2\right)\hat{\phi}(x)\right) d^3x$$

- But how do $\hat{\phi}(x)$ and $\hat{\pi}(x)$ act on wave functionals $\Psi[\phi,t]$?

- A valid representation of $[\hat{\phi}(x),\hat{\pi}(x')] = i\delta^3(x-x')$ (Exercise: check) is:

$$\hat{\phi}(x) . \Psi[\phi,t] = \phi(x)\Psi[\phi,t]$$

Analogous to: $\hat{q}_a . \Psi(q,t) = q_a \Psi(q,t)$

$$\hat{\pi}(x) . \Psi[\phi,t] = -i\frac{\delta}{\delta\phi(x)}\Psi[\phi,t]$$

$\hat{p}_a . \Psi(q,t) = -i\frac{\partial}{\partial q_a}\Psi(q,t)$

functional derivative, as in variational principle used to derive Euler Lagrange equations.

- Therefore:

$$\hat{H} = \int \frac{1}{2}\left(-\frac{\delta^2}{\delta\phi^2(x)} + \phi(x)\left(-\Delta + m^2\right)\phi(x)\right)$$

inconvenient

- It is more convenient to use infrared-regularized momentum space:

- We now need to represent

$$[\hat{\phi}_k, \hat{\pi}_{k'}] = i\delta_{k,-k'}$$

on the wave functionals $\Psi[\tilde{\phi}, t]$. ($\tilde{\phi}_k$ is Fourier transform of $\phi(x)$)

- As you should verify, this works:

$$\hat{\phi}_k \cdot \Psi[\tilde{\phi}, t] = \tilde{\phi}_k \Psi[\tilde{\phi}, t]$$

$$\hat{\pi}_k \cdot \Psi[\tilde{\phi}, t] = -i \frac{\partial}{\partial \phi_{-k}} \Psi[\tilde{\phi}, t]$$

Note: Ordinary derivatives here because set of variables $\{\tilde{\phi}_k\}$ is discrete, since $k = \frac{2\pi}{L}(n_1, n_2, n_3)$, $\vec{n} \in \mathbb{Z}^3$.

Schrödinger equation:

$i\partial_t |\Psi\rangle = \hat{H} |\Psi\rangle$ becomes:

$$i\partial_t \Psi[\tilde{\phi}, t] = \sum_k \frac{1}{2}\left(-\frac{\partial}{\partial \tilde{\phi}_k}\frac{\partial}{\partial \tilde{\phi}_{-k}} + (k^2 + m^2)\tilde{\phi}_k \tilde{\phi}_{-k}\right) \Psi[\tilde{\phi}, t]$$

Recall: For QM harm. osc., ground state Schrödinger wave function is:

$$\Psi(x,t) = N e^{-\frac{1}{2}\omega x^2 - i\omega t}$$

Exercise: check it. Can you solve for excited states?

$\Rightarrow$ Ground state solution in QFT reads, similarly:

$$\Psi[\tilde{\phi}, t] = N e^{-\sum_k \frac{1}{2}(\omega_k \tilde{\phi}_k \tilde{\phi}_{-k} - i\omega_k t)}$$

$\omega_k = (\vec{k}^2 + m^2)^{1/2}$

Exercise: verify

... which we had already seen before.

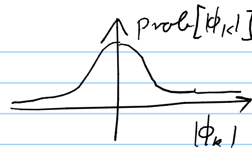
Generic wave functionals

- Assume the system is in a state, $|\alpha\rangle$, other than $|\Psi_0\rangle$.

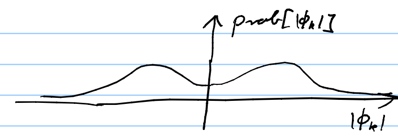
  $\Rightarrow$ For at least some modes oscillators, $|\alpha\rangle$ is not the ground state.

- But if an oscillator is excited, then its wave function spreads out – classically its amplitude of oscillation would increase.

  $\Rightarrow$ If a mode k is excited then the prob. distribution of the ϕ_k spreads:



ground state



example of excited state

- The more a mode k is excited, the more likely is a measurement of $\hat{\phi}_k$ to yield a ϕ_k with large modulus $|\phi_k|$.

$\Rightarrow$ If, e.g., a mode k is very highly excited then $|\phi_k|$ is likely very large,

$\Rightarrow$ measurement of $\hat{\phi}(x)$ likely yields $\phi(x)$ which shows a plane wave in the direction k – on top of the usual quantum fluctuations.

$\rightsquigarrow$ see project description.