Information-Theoretic Natural Ultraviolet Cutoff for Spacetime

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Fields in spacetime could be simultaneously discrete *and* continuous, in the same way that information can. It has been shown that the amplitudes $\phi(x_n)$ that a field takes at a generic discrete set of points x_n can be sufficient to reconstruct the field $\phi(x)$ for all x, namely, if there exists a certain type of natural ultraviolet (UV) cutoff in nature, and if the average spacing of the sample points is at the UV-cutoff scale. Here, we generalize this information-theoretic framework to spacetimes themselves. We show that samples taken at a generic discrete set of points of a Euclidean-signature spacetime can allow one to reconstruct the shape of that spacetime everywhere, down to the cutoff scale. The resulting methods could be useful in various approaches to quantum gravity.

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At the heart of any candidate theory of quantum gravity is an attempt to describe the structure of spacetime at the Planck scale. The problem is hard because general relativity and quantum theory provide seemingly contradictory indications. While general relativity describes spacetime as a manifold, quantum field theories appear to be well defined only if spacetime is discrete.

In this context, it has been proposed that spacetime could be simultaneously discrete and continuous, in the same way that information can [1]. The underlying mathematical structure, Shannon sampling theory [2], is at the heart of information theory and is of ubiquitous use in communication engineering and signal processing. Sampling theory explains that any signal $\phi(t)$ with a band limit Ω can be reconstructed perfectly for all t from the knowledge of the samples $\phi(t_n)$ at sample points t_n whose average density (technically, the Beurling density) is at least 2Ω. The reconstruction of $\phi(t)$ is numerically most stable when the samples are taken equidistantly, with spacing $t_{n+1} - t_n = (2\Omega)^{-1}$, in which case the reconstruction formula reads $\phi(t) = \sum_n \mathrm{sinc}[2(t - t_n)\Omega]\phi(t_n)$. For later reference, we note that then also, $\sum_n \phi(t_n)\psi(t_n) = 2\Omega \int dt \phi(t)\psi(t)$, which is useful to solve hard-to-sum series (e.g., in analytic number theory): view the terms of the series in question as samples of bandlimited functions, rewrite the series as an integral, then apply integration tools such as integration by parts or contour integration.

The idea that physical fields could possess the sampling property was first proposed in [1], where it was shown that fields possess the sampling property if the uncertainty relations are modified in the ultraviolet so that there is formally a finite lower bound Δx_{min} on the uncertainty in position. Such uncertainty relations have indeed arisen in various studies of quantum gravity and string theory; see, e.g., [3]. This UV cutoff has then been applied, in particular, to the spacelike hypersurfaces in inflationary cosmology. Possible signatures in the scalar and tensor spectra of the cosmic microwave background have been calculated; see, e.g., [4].

Sampling theory of fields.—Let us briefly review the generalization of sampling theory to physical fields in curved Euclidean-signature spacetimes, such as the fields that are being summed over in the path integral of Euclidean quantum field theory (QFT), [5,6].

To this end, consider a spacetime described by a compact smooth Riemannian manifold. For simplicity, let us assume it has no boundary. For the covariant inner product of fields on the manifold we use the usual bra-ket notation inspired by first quantization: $(\phi|\psi) = \int d^d x \sqrt{|g|}\phi(x)\psi(x)$, so that one has, for example: $\phi(x) = (x|\phi)$. We use the sign convention in which the spectrum of the Laplacian is positive and we choose $c = \hbar = G = 1$. The Laplacian is self-adjoint, with $\Delta v_{\lambda_i} = \lambda_i v_{\lambda_i}$ solved by normalizable eigenfunctions with discrete eigenvalues. The generalization of the assumption of band limitation is the assumption that there exists a natural hard UV cutoff Λ of the spectrum of the Laplacian, with Λ, for example, at the Planck scale. In QFT, the space of fields $\mathcal{F}$ that is being integrated over in the path integral, is then spanned by the eigenfunctions v_{λ_i} of the Laplacian whose eigenvalues λ_i are below the cutoff $\lambda_i < \Lambda$. Let P denote the projector onto $\mathcal{F}$ and let us denote the Laplacian restricted to $\mathcal{F}$ by $\Delta_c = \Delta|_{\mathcal{F}}$. The fields $|\phi) \in \mathcal{F}$ that occur in the path integral obey $\phi(x) = (x|\phi) = (x|P|\phi)$. Notice that this means that the point-localized fields $|x)$ are now indistinguishable from the fields $P|x)$ in which wavelengths shorter than the cutoff scale are removed. Intuitively, this expresses a minimum length uncertainty principle.

Since the Laplacian's spectrum does not possess accumulation points, the dimension N of the space of fields is finite, $\dim(\mathcal{F}) = N$. It was shown that therefore any field $\phi(x) \in \mathcal{F}$ can be reconstructed everywhere if known only on N generic points of the manifold. The fields, actions, and equations of motion therefore possess a representation on the smooth spacetime manifold as well as equivalently also on any lattice of N generic points. Under mild conditions, Weyl's asymptotic formula was shown to imply that as the infrared (IR) cutoff is removed by letting the volume of the manifold diverge, $V \to \infty$, one has $N \to \infty$ such that the density of samples necessary for reconstruction, N/V, indeed stays finite [6].

Sampling theory of spacetime.—The aim in this Letter is to generalize sampling theory to spacetime itself.

Assuming the natural hard UV cutoff above, can the shape (curvature and global topology) of a Euclidean-signature spacetime be reconstructed everywhere from suitable samples taken at a discrete set of points?

For most purposes, a spacetime's shape is best described in terms of the affine connection. Here, however, a different description appears more useful. Let us recall a comment by Einstein [7], who pointed out that the nontrivial shape of a manifold manifests itself not only in the nontriviality of the parallel transport of tensors. Crediting Helmholtz, Einstein emphasized that the shape of a manifold can also be thought of in terms of the nontriviality of the mutual distances among points: In d-dimensional flat space, consider M points. In Cartesian coordinates, the points possess Md coordinates $x_i^{(n)}$ with $n = 1, \ldots, M$ and $i = 1, \ldots, d$. By Pythagoras, the $M(M-1)/2$ mutual distances $s_{n,n'}$ obey the equations $s_{n,n'}^2 = \sum_{i=1}^d (x_i^{(n)} - x_i^{(n')})^2$. If $M > 2d + 1$, the Md coordinates can be eliminated in these $M(M-1)/2$ equations, to leave $M(M-1)/2 - Md$ nontrivial equations that must hold among the mutual distances $s_{n,n'}$ if the manifold is indeed flat. If the manifold is curved this manifests itself in the way in which these equations are violated.

Let us try, therefore, to reconstruct the shape of a spacetime of finite volume by sampling at a sufficient number N of generic points a quantity that is closely related to their mutual distances. To this end, we sample the propagator, or correlator, $G(x^{(n)}, x^{(n')})$, of a scalar field for each pair of the N chosen points. Generally, the larger the distance between $x^{(n)}$ and $x^{(n')}$, the smaller is the correlator. For a free scalar we have, e.g., $G(x^{(n)}, x^{(n')}) = (x^{(n)}|P(\Delta + m^2)^{-1}P|x^{(n')})$. Indeed, the knowledge of the $N(N-1)/2$ matrix elements $G(x^{(n)}, x^{(n')})$ suffices to reconstruct the shape of the spacetime up to the UV-cutoff scale. To see this, we note first that the matrix $[G(x^{(n)}, x^{(n')})]_{nn'}$ represents the correlator $(\Delta_c + m^2)^{-1}$ in a basis, namely, the basis $\{P|x^{(n)})\}$, which means that we can determine its eigenvalues. Since the correlator is diagonal in the same basis as the Laplacian Δ_c, we also obtain the spectrum of Δ_c.

Crucially now, the eigenvalues of Δ_c provide us with the shape of the spacetime from large length scales down to the cutoff scale. To see this, let us recall key results of the discipline of spectral geometry which studies the relationship between the shape of a manifold (or domain) and the spectrum of its Laplacian or Dirac operator. (Note that spectral geometry thereby naturally combines functional analysis and differential geometry, i.e., the mathematical languages of quantum theory and general relativity.) In particular, isospectral manifolds are generally also isometric, though there are exceptions. One cannot always "hear the shape of a drum"; see, e.g., [8]. Even pairs of isospectral but nonisometric manifolds that are compact and simply connected have been constructed [9]. It is known, however, that the eigenvalues change continuously as a function of the shape of the manifold. Also, the eigenvalues are nondegenerate for generic manifolds, and a manifold can have degenerate eigenvalues only if it possesses a continuous group of isometries [10].

For our purposes, we are led to consider classes of manifolds whose Laplacians share the same eigenvalues (and their multiplicities) only up to the cutoff Λ, and which we may therefore call Λ isospectral. This is because the samples of the matrix elements of the correlator $[G(x^{(n)}, x^{(n')})]_{nn'}$ determine only the N eigenvalues of the Laplacian Δ_c. The eigenvalues that the full Laplacian Δ possesses beyond the cutoff remain undetermined. This tells us what the UV cutoff means for the shape of spacetime itself. The cutoff does not directly mean a cutoff for the curvature, for example. Instead, the fact that the eigenvalues of the Laplacian beyond the cutoff remain undetermined by any measurement possible means that all Λ-isospectral manifolds are physically indistinguishable and thus equivalent. When referring to a "spacetime with UV cutoff," specified by N eigenvalues, $\lambda_1 \leq \ldots \leq \lambda_N < \Lambda$, we will therefore henceforth mean an equivalence class of Λ-isospectral manifolds.

Intuitively, the higher the eigenvalues of the Laplacian the higher the "squared momentum" that they represent, and therefore the smaller the wrinkles which they determine in the manifold. The eigenvalues up to Λ essentially determine the shape of a spacetime with UV cutoff from large scales down to lengths as small as the cutoff scale. The undetermined eigenvalues beyond the cutoff would describe wrinkles on length scales smaller than the cutoff scale. At distances smaller than the cutoff scale the shape of a spacetime with UV cutoff is not determined.

This can be viewed as a matter of representation theory. In general relativity, the choice of coordinate system is merely a choice of representation for an underlying Riemannian manifold. With the UV cutoff, even the choice of Riemannian manifold is merely a choice of representation for an underlying "spacetime with UV cutoff" that is defined through the first N eigenvalues of the Laplacian.

To be precise, however, the picture is slightly more subtle. A theory may contain additional fields with interactions that allow one to physically distinguish among certain Λ-isospectral manifolds. At the very least, we have to divide each equivalence class of Λ-isospectral manifolds into subequivalence classes of manifolds that are continuously deformable into each other within their class of Λ-isospectral manifolds. This is because at least for those subclasses, and possibly also for subsubclasses within them, we can define what we may call "geometric quantum numbers" that distinguish them and that could be measurable in the full theory. Consider, for example, a manifold in the shape of a potato's surface, with N points singled out. The same N points on the manifold with the shape of the mirror-imaged potato clearly possess the same correlators. Thus, the two potato surfaces, similar to enantiomers in chemistry, are Λ sospectral. However, if the action contains a parity-breaking interaction, such as the weak interaction, then the two manifolds become distinguishable. Apart from parity, also the dimension of the manifold can be viewed as a geometric quantum number, measurable, e.g., through interactions involving tensors (whose dimensions indicate the manifold's dimension). Indeed, Λ-isospectral manifolds of different dimensions cannot be continuously deformed into another [10]. Note that the scaling of the Laplacian's spectrum for asymptotically large eigenvalues is in one-to-one correspondence to the manifold's dimension [11]. It should be interesting to study the set of possible geometric quantum numbers, and their relation to cohomology, by methods similar to those used to construct isospectral nonisometric manifolds; see, e.g., [9].

A "spacetime with UV cutoff" is, therefore, an equivalence class of manifolds that are Λ isospectral and possess the same geometric quantum numbers. An intriguing possibility is that it may not be necessary to keep track of geometric quantum numbers as variables that are separate from the spectrum after all, namely, when working with the full Laplacian, $d\delta + \delta d$, on all differential forms, and the Dirac operator. Their spectra may well include all information about the geometric quantum numbers. For example, the multiplicity of the eigenvalue 0 of Δ on p forms yields the manifold's pth Betti number [11], and the largest value of p yields its dimension.

Let us summarize the sampling and reconstruction for both spacetime and field. Abstractly, a spacetime is specified by the eigenvalues $\lambda_1, \ldots, \lambda_N$ of Δ_c. A field $|\phi)$ on the spacetime is a vector in the N-dimensional Hilbert space on which Δ_c acts, conveniently specified through its coefficients ϕ_i in an ON eigenbasis $\{|v_{\lambda_i})\}$ of Δ_c, as $|\phi) = \sum_{i=1}^N \phi_i |v_{\lambda_i})$. Now consider a continuous representation of the spacetime as a manifold that possesses the right geometric quantum numbers, such as the dimension, and whose Laplacian Δ possesses the spectrum of Δ_c up to Λ. Choosing coordinates, the eigenvectors of Δ_c are represented as eigenfunctions $v_{\lambda_i}(x)$ of Δ. The field is represented by the function $\phi(x) = \sum_{i=1}^N \phi_i v_{\lambda_i}(x)$. We obtain a lattice representation by choosing N generic points $x^{(1)}, \ldots, x^{(N)}$ at which we sample and record the matrix of correlators $G(x^{(i)}, x^{(j)})$ and the field's amplitudes $\phi(x^{(i)})$. From these data we can fully reconstruct the $\lambda_1, \ldots, \lambda_N$ and $\phi_1, \ldots, \phi_N$ that abstractly define the spacetime and field. First, the diagonalization of the matrix of correlators yields the eigenvalues of Δ_c. To obtain the coefficients $\phi_j = (v_{\lambda_j}|\phi)$ we insert a resolution of the identity in $(x^{(n)}|\phi) = \sum_{j=1}^N (x^{(n)}|v_{\lambda_j})(v_{\lambda_j}|\phi)$, i.e., $\phi(x^{(n)}) = \sum_j E_{nj}\phi_j$. The change of basis matrix $E_{nj} = (x^{(n)}|v_{\lambda_j})$ is known from the diagonalization of the matrix of correlator samples. Since the $\phi(x^{(i)})$ are known samples, we obtain $\phi_i = \sum_j (E_{ij})^{-1}\phi(x^{(j)})$. Thus, from the samples of field amplitudes and correlators, we have obtained the spacetime in terms of the eigenvalues of its Laplacian Δ_c and the field as a vector in the N-dimensional vector space on which Δ_c acts. At this point we are free to represent the abstract spacetime by the same or any other member of its equivalence class of Λ-isospectral manifolds, choose coordinates, and express the field as an explicit function. Note that the choice of real-valued orthonormalized eigenfunctions, while normally ambiguous up to a factor of -1, is here fixed by continuity. The formalism establishes, therefore, an equivalence between discrete and continuous representations of spacetimes and fields.

Note that the reconstruction of the spacetime and the field is possible even when the sample points x_n are chosen close together, possibly leaving large regions without samples. As mentioned, also in Shannon sampling the reconstruction from irregularly-spaced samples is possible but requires increased numerical precision. This can be understood by considering Shannon's channel capacity formula, $C = B\log(1 + S/N)$ and the phenomenon of superoscillations [2]. While holding the overall information density C fixed, the sample density, or Nyquist rate B, can be locally lowered, at the cost of needing an exponentially higher signal to noise ratio S/N, which represents the reconstruction instability. In our case here, we notice that if the x_n are chosen close, the basis vectors $P|x_n)$ acquire significant overlap and thus form small angles in the Hilbert space $\mathcal{F}$. Thus, the condition number of the change of basis matrix E to the ON eigenbasis of Δ_c deteriorates, indicating the need for increased numerical precision.

A simple example of a partition function.—Consider a matter action of the form $S_{\text{matter}} = \int d^n x\sqrt{|g|}[\frac{1}{2}\phi(x)(\Delta + m^2)\phi(x) + J(x)\Phi(x)]$. Representation independently, the action reads $S_{\text{matter}} = \frac{1}{2}(\phi|\Delta + m^2|\phi) + (J|\phi) = \text{Tr}[\frac{1}{2}\times(\Delta + m^2)|\phi)(\phi| + |J)(\phi|]$. Representing the action in an ON eigenbasis of the Laplacian, we have $S_{\text{matter}} = \sum_{i=1}^N \frac{1}{2}\phi_i(\lambda_i + m^2)\phi_i + J_i\phi_i$.

The action already contains the degrees of freedom λ_i, which describe the shape of the spacetime. However, an action is still needed for the degree of freedom N, which describes the overall size of the system. The simplest choice is $S_{\text{size}} = \alpha N = \alpha\,\text{Tr}(1)$, where α remains to be determined. We obtain $S_{\text{total}} = \text{Tr}[\alpha 1 + \frac{1}{2}(\Delta + m^2)|\phi)\times(\phi| + |J)(\phi|]$. To understand S_{size}, let us represent the spacetime as a manifold. Then, N, being a scalar, is expressible as an integral over the curvature scalars. Indeed, in four dimensions, [12]:

$$N = \frac{1}{16\pi^2}\int d^4x\sqrt{|g|}\Big\{\frac{\Lambda^2}{2} + \frac{\Lambda}{6}R + \frac{1}{180}\Big(R^{\mu\nu\rho\epsilon}R_{\mu\nu\rho\epsilon} - R_{\mu\nu}R^{\mu\nu} + 6\Delta R - \frac{5}{2}R^2\Big) + O(\Lambda^{-1})\Big\}. \tag{1}$$

The first term yields a cosmological constant. It implies that the average density of sample points needed for the reconstruction of the field and spacetime obeys $N/V = \Lambda^2/32\pi^2$, except for corrections due to the curvature terms. Curvature can therefore be viewed as a spatial modulation of the average sample density, or density of degrees of freedom. The second term in S_{size} becomes the Einstein action after we set $\alpha = 6\pi/\Lambda$.

We could now represent the spacetimes and fields in the formal partition function $Z[J] = \int e^{-S_{\text{total}}} D[\phi]D[g]$ as concrete manifolds and concrete functions on those manifolds. This would yield an unwieldy expression whose evaluation would be plagued, as usual, by the need to mod out physically equivalent configurations. Alternatively, the sampling-theoretic view suggests working with the well-defined partition function

$$Z[J] = \sum_{N=1}^{\infty} \int D[\phi] \times \int D[\lambda] e^{-\text{Tr}((6\pi/\Lambda)+(1/2)(\Delta+m^2)|\phi)(\phi|+|J)(\phi|)}, \quad (2)$$

which reads in the eigenbasis of the Laplacian

$$Z[J] = \sum_{N=1}^{\infty} \frac{1}{N!} \int_0^{\Lambda} d\lambda_1 \ldots \int_0^{\Lambda} d\lambda_N \times \int_{-\infty}^{\infty} d\phi_1 \ldots \int_{-\infty}^{\infty} d\phi_N \exp\left[-\frac{6\pi N}{\Lambda} - \sum_{i=1}^{N}\left(\frac{1}{2}(\lambda_i + m^2)\phi_i^2 - J_i\phi_i\right)\right].$$

Discussion.—Note that we here tentatively assumed that it is necessary to sum over all discrete spectra (up to Λ), irrespective of what type of Riemannian manifold, if any, they correspond to. Also, in the full theory, there could be additional terms that induce the cutoff dynamically, for example [5], (counter-) terms that form a power series in the Laplacian, $(\phi|\sum_r c_r \Delta^r|\phi) = \text{Tr}(\sum_r c_r \Delta^r|\phi)(\phi|)$, whose radius of convergence is Λ. This removes fields that contain components beyond the cutoff by letting their Boltzmann factor vanish. In general, the terms responsible for the cutoff may of course not be quadratic in the fields. Further, it may be necessary to handle manifolds with infinite volume and a continuous spectrum. In this case, one may rescale Δ_c as $N \to \infty$, to keep its spectrum discrete and therefore indicative of the manifold's shape.

Since any candidate quantum gravity theory must recover QFT for sufficiently large length scales, we discussed how the sampling-theoretic natural UV cutoff would impact the QFT path integral. The new framework for the sampling and reconstruction of spacetimes and fields could be useful, however, also beyond QFT in various studies of quantum gravity, in particular, in studies in which spacetime is modeled as discrete; see, e.g., [13]. There, the new sampling methods could be used to give discrete structures a continuous representation. This could establish and stabilize the effective dimension of a lattice and it could make it unnecessary to take a continuum limit. It could also serve as a mere mathematical tool in these theories, for example, to rewrite hard-to-sum series as integrals, or to rewrite hard-to-solve finite difference equations as differential equations that are easier to handle. Work is in progress on linking manifold and graph Laplacians, and also on reconstructing the shape of Lorentzian manifolds from the mutual distances of events as measured through two-point functions. Also, formulations of general relativity in terms of the eigenvalues of the Dirac operator have been discussed, e.g., in [14]. It should be interesting to transfer results of those works into the information-theoretic framework here.

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