expressions that involve all those positions and momentum variables? To this end, we need to define what the Poisson brackets in between positions and momenta of different particles should be. They are defined to be simply zero. Therefore, to summarize, we define the basic Poisson brackets of n masses as

$$\{x_i^{(r)}, p_j^{(s)}\} = \delta_{i,j}\ \delta_{r,s} \tag{2.16}$$

$$\{x_i^{(r)}, x_j^{(s)}\} = 0 \tag{2.17}$$

$$\{p_i^{(r)}, p_j^{(s)}\} = 0 \tag{2.18}$$

where $r, s \in \{1, 2, ..., n\}$ and $i, j \in \{1, 2, 3\}$. The evaluation rules of Eqs.2.9-2.13 are defined to stay just the same.

Exercise 2.6 *Mathematically, the set of polynomials in positions and momenta is an example of what is called a Poisson algebra. A general Poisson algebra is a vector space with two extra multiplications: One multiplication which makes the vector space into an associative algebra, and one (non-associative) multiplication* $\{,\}$*, called the Lie bracket, which makes the vector space into what is called a Lie algebra. If the two multiplications are in a certain sense compatible then the set is said to be a Poisson algebra. Look up and state the axioms of a) a Lie algebra, b) an associative algebra and c) a Poisson algebra.*

2.3.3 The Hamilton equations

Let us recall why we introduced the Poisson bracket: A technique that uses the Poisson bracket is supposed to allow us to *derive* all the differential equations of motion of a system from the just one piece of information, namely from the expression of the total energy of the system, i.e., from its Hamiltonian.

To see how this works, let us consider an arbitrary polynomial f in terms of the positions and momentum variables $x_i^{(r)}, p_j^{(s)}$ of the system in question, for example, something like $f = 7x_2^{(3)}\left(x_3^{(1)}\right)^3 - 2\cos(4t^2)(p_1^{(1)})^7 + 3/2$. This f depends on time for two reasons: There is an explicit dependence on time through the cosine term, and there is an implicit dependence on time because the positions and momenta generally depend on time. According to Hamilton's formalism, the equation of motion for f is then given by:

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \tag{2.19}$$

Here, the notation $\partial f/\partial t$ denotes differentiation of f with respect to only its explicit time dependence. In the example above it is the derivative of the time dependence in the term $\cos(4t^2)$.

Eq.2.19 is a famous equation which is called the Hamilton equation. Why is it famous? If you know how to evaluate Poisson brackets then the Hamilton equation

Eq.2.19 encodes for you all of classical mechanics! Namely, given H, equation Eq.2.19 yields the differential equation of motion for any entity f by the simple procedure of evaluating the Poisson bracket on its right hand side.

If f is dependent on time only through x and p (say if we choose for f a polynomial in x and p's with constant coefficients) then $\partial f/\partial t = 0$ and Hamilton's equation simplifies to:

$$\frac{d}{dt} f \;=\; \{f, H\} \tag{2.20}$$

In the remainder of this book, unless otherwise specified, we will always choose our functions f, g, h to have no explicit dependence on time, i.e., they will depend on time only implicitly, namely through the time dependence of the x and p's. In particular, the most important choices for f are of this kind: $f = x_i^{(r)}$ or $f = p_i^{(r)}$. For these choices of f we immediately obtain the fundamental equations of motion:

$$\frac{d}{dt} x_i^{(r)} \;=\; \{x_i^{(r)}, H\} \tag{2.21}$$

$$\frac{d}{dt} p_i^{(r)} \;=\; \{p_i^{(r)}, H\} \tag{2.22}$$

Here is a concrete example: A single free particle of mass m possesses only kinetic energy. Its Hamiltonian is:

$$H = \sum_{j=1}^{3} \frac{p_j^2}{2m} \tag{2.23}$$

By using this H in Eqs.2.21,2.22, we obtain the following equations of motion for the positions and momenta:

$$\frac{d}{dt} x_i \;=\; \left\{ x_i \;,\; \sum_{j=1}^{3} \frac{p_j^2}{2m} \right\} \;=\; \frac{p_i}{m} \tag{2.24}$$

and

$$\frac{d}{dt} p_i \;=\; \left\{ p_i \;,\; \sum_{j=1}^{3} \frac{p_j^2}{2m} \right\} \;=\; 0 \tag{2.25}$$

They agree with what was expected: $p_i = m\dot{x}_i$ and $\ddot{x}_i = 0$, where the dot indicates the time derivative. For another example, consider again the system of two point masses m_1, m_2 which are connected by a spring with spring constant k. Its Hamiltonian H was given in Eq.2.4. By using this H in Eqs.2.21,2.22 we should now be able to derive the system's equations of motion (as given in Eqs.2.1-2.3). Indeed:

$$\frac{d}{dt} x_i^{(r)} \;=\; \{x_i^{(r)}, H\} \tag{2.26}$$

$$\;=\; \frac{p_i^{(r)}}{m_r} \tag{2.27}$$

$$\frac{d}{dt}\, p_i^{(1)} = \{p_i^{(1)}, H\} \tag{2.28}$$
$$= -k(x_i^{(1)} - x_i^{(2)}) \tag{2.29}$$

$$\frac{d}{dt}\, p_i^{(2)} = \{p_i^{(2)}, H\} \tag{2.30}$$
$$= -k(x_i^{(2)} - x_i^{(1)}) \tag{2.31}$$

Let us omit the proof that Hamilton's formulation of classical mechanics always yields the same equations of motion as Newton's.

Exercise 2.7 *Consider $f = gh$, where g and h are some polynomial expressions in the position and momentum variables. There are two ways to calculate df/dt: Either we use the Leibnitz rule, i.e., $\dot{f} = \dot{g}h + g\dot{h}$, and apply Eq.2.20 to both $\dot{g}$ and $\dot{h}$, or we apply Eq.2.20 directly to gh and use the product rule (Eq.2.12) for Poisson brackets. Prove that both methods yield the same result.*

This exercise shows that a property of the derivative on the left hand side of Eq.2.20 determines a rule for how the Poisson bracket had to be defined. In fact, such requirements of consistency are the main reason why the Poisson bracket is defined the way it is.

Exercise 2.8 *Use Eq.2.13 to prove that:*

$$\frac{d}{dt}\,\{f, g\} = \{\dot{f}, g\} + \{f, \dot{g}\} \tag{2.32}$$

2.3.4 Symmetries and Conservation laws

Our reason for reviewing the Hamiltonian formulation of mechanics is that it will be useful for the study of quantum mechanics. Before we get to that, however, let us ask why the Hamiltonian formulation of mechanics was useful for classical mechanics. It was, after all, developed more than half a century before quantum mechanics.

Sure, it was fine to be able to derive all the differential equations of motion from the one unifying equation. Namely, if for simplicity, we are now assuming that $f(x, p)$ is chosen not to have any explicit dependence on time, the universal Hamilton equation which expresses the differential equation of all variables f, is:

$$\dot{f} = \{f, H\} \tag{2.33}$$

Ultimately, however, one obtains this way just the same equations of motion as Newton's methods would yield. Was there any practical advantage to using Hamilton's formulation of mechanics? Indeed, there is an important practical advantage: The main advantage of Hamilton's formulation of mechanics is that it gives us powerful methods for studying conserved quantities, such as the energy or angular momentum. In general, for complicated systems (such as planetary systems, for example), there can be much more complicated conserved quantities than these. To know conserved quantities usually significantly helps in solving the dynamics of complicated systems. This feature of Hamilton's formulation of mechanics will carry over to quantum mechanics, so studying it here will later help us also in quantum mechanics.

Consider a polynomial f in x and p's with constant coefficients. Then, $\partial f/\partial t = 0$ and Eq.2.33 applies. We can easily read off from Eq.2.33 that *any such f is conserved in time if and only if its Poisson bracket with the Hamiltonian vanishes:*

$$\{f, H\} = 0 \quad \Rightarrow \quad \dot{f} = 0 \tag{2.34}$$

Consider, for example, a free particle. Its Hamiltonian is given in Eq.2.23. We expect of course that its momenta p_i are conserved. Indeed:

$$\dot{p}_i = \left\{ p_i, \sum_{j=1}^{3} p_j^2/2m \right\} = 0 \tag{2.35}$$

For another example, consider a system whose Hamiltonian is any polynomial in x's and p's with constant coefficients. The proof that this system's energy is conserved is now fairly trivial to see (using Axiom Eq.2.9):

$$\dot{H} = \{H, H\} = 0 \tag{2.36}$$

I mentioned that in order to be able to find solutions to the equations of motion of complicated real-life systems it is often crucial to find as many conserved quantities as possible. Here is an example. Consider a 3-dimensional isotropic (i.e., rotation invariant) harmonic oscillator. Because of its symmetry under rotations, it angular momentum is conserved. But this oscillator has actually a much larger symmetry and therefore more conserved quantities. This is because a harmonic oscillator, being of the form $x^2 + p^2$ also possesses rotation symmetry in phase space. I will here only remark that this means that the 3-dimensional isotropic harmonic oscillator possesses $SO(3)$ rotational symmetry as well as a larger $SU(3)$ symmetry.

Powerful methods for discovering symmetries and constructing the implied conserved quantities for arbitrary systems have been developed on the basis of Eq.2.33 and the Poisson bracket. A key technique is that of so-called canonical transformations, i.e., of changes variables for which the Poisson brackets remain the same. You can find these methods in classical mechanics texts under the keywords "canonical transformations" and "Hamilton Jacobi theory".

In fact, Poisson bracket methods reveal a very deep one-to-one correspondence between conserved quantities and so-called symmetries. For example, the statement that an experiment on a system gives the same result no matter when we perform the experiment, is the statement of a "symmetry" which is called time-translation invariance symmetry. In practice, it means that the Hamiltonian of the system does not explicitly depend on time: $\partial H/\partial t = 0$. As we just saw, this implies energy conservation: $dH/dt = 0$.

Similarly, the statement that an experiment on a system gives the same result wherever we perform the experiment is the statement of space-translation symmetry. It implies and is implied by momentum conservation. Further, the statement that an experiment on a system gives the same result whatever the angular orientation of the experiment is the statement of rotation symmetry. It implies and is implied by angular momentum conservation.

These are examples of the Noether theorem, of Emmi Noether. Her theorem plays a crucial role both in practical applications, and in fundamental physics[6]. We will later come back to Noether's theorem.

Exercise 2.9 *Show that if H is a polynomial in the positions and momenta with arbitrary (and possibly time-dependent) coefficients, it is true that $dH/dt = \partial H/\partial t$.*

Exercise 2.10 *Consider the system with the Hamiltonian of Eq.2.4. Show that the total momentum is conserved, i.e., that $p_i^{(1)} + p_i^{(2)}$ is conserved for all i.*

2.3.5 A representation of the Poisson bracket

In principle, we can evaluate any Poisson bracket $\{f, g\}$ by using the rules Eqs.2.6-2.12 if, as we assume, f and g are polynomials or well-behaved power series in the position and momentum variables. This is because the product rule allows us to break Poisson brackets that contain polynomials into factors of Poisson brackets that contain polynomials of lower degree. Repeating the process, we are eventually left with having to evaluate only Poisson brackets of linear terms, which can easily be evaluated using the first set of rules.

This is all good and fine but when f or g contain high or even infinite powers of the position and momentum variables, then the evaluation of the Poisson bracket $\{f, g\}$ can become rather tedious and cumbersome.

For practical purposes it is of interest, therefore, to have a shortcut for the evaluation of Poisson brackets. Indeed, for complicated f and g, the Poisson bracket $\{f, g\}$ can be evaluated usually faster by the following formula:

$$\{f, g\} = \sum_{r=1}^{n} \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i^{(r)}} \frac{\partial g}{\partial p_i^{(r)}} - \frac{\partial f}{\partial p_i^{(r)}} \frac{\partial g}{\partial x_i^{(r)}} \right) \tag{2.37}$$

[6]Mathematically, symmetries are described as groups (for example, the composition of two rotations yields a rotation and to every rotation there is an inverse rotation). In elementary particle physics, symmetry groups are one abstraction level higher than Hamiltonians: It has turned out that the complicated Hamiltonians which describe the fundamental forces, i.e., the electromagnetic, weak and strong force, are essentially derivable as being the simplest Hamiltonians associated with with three elementary symmetry groups.

Exercise 2.11 *Use Eq.2.37 to evaluate* $\{x^8p^6, x^3p^4\}$.

Exercise 2.12 *Show that Eq.2.37 is indeed a representation of the Poisson bracket, i.e., that it always yields the correct answer. To this end, check that it obeys Eqs.2.9-2.13 and Eqs.2.16-2.18 (except: no need to check the Jacobi identity as that would be straightforward but too tedious).*

Exercise 2.13 *Find the representation of the Hamilton equations Eq.2.19 and Eqs.2.21, 2.22 obtained by using Eq.2.37.*

Remark: Some textbooks start with these representations of the Hamilton equations, along with the representation Eq.2.37 of the Poisson bracket - without reference to the Hamilton equations' more abstract origin in Eq.2.19 and Eqs.2.21, 2.22. This is unfortunate because those representations using Eq.2.37 do not carry over to quantum mechanics, while the more abstract equations Eq.2.19 and Eqs.2.21, 2.22 will carry over to quantum mechanics unchanged, as we will see!

2.4 Summary: The laws of classical mechanics

We already discussed that quantum mechanics must have strong similarities with classical mechanics, since it must reproduce all the successes of classical mechanics. This suggested that the laws of quantum mechanics might be a slight modification of Newton's laws which would somehow contain Planck's constant h. Since this did not work, we reformulated the laws of classical mechanics on a higher level of abstraction, namely in Hamilton's form. Before we now try to guess the laws of quantum mechanics, let us restate Hamilton's formulation of classical mechanics very carefully:

The starting point is the energy function H of the system in question. It is called the Hamiltonian, and it is an expression in terms of the position and momentum variables of the system. Then, assume we are interested in the time evolution of some quantity f which is also a polynomial in the x and p's (say with constant coefficients). Then we can derive the equation of motion for f through:

$$\frac{d}{dt} f \;=\; \{f, H\} \tag{2.38}$$

In particular, f can be chosen to be any one of the position and momentum variables of the system, and we obtain their equations of motion as Eqs.2.21,2.22. In order to obtain explicit differential equations from Eqs.2.38,2.21,2.22 we evaluate the Poisson bracket on its right hand side. To this end, we use the definitions Eqs.2.6-2.13. The so-obtained differential equations are then solved to obtain the positions $x_i^{(r)}(t)$ and momenta $p_i^{(r)}(t)$ as functions of time.

We note that the Poisson bracket which is defined by the axioms Eqs.2.6-2.12 possesses an often convenient explicit representation through Eq.2.37. We need to keep in mind, however, that Eq.2.37 merely provides a convenient shortcut for evaluating the Poisson bracket. This shortcut only works in classical mechanics. In quantum mechanics, there will also be a representation of the Poisson bracket but it will look very different from Eq.2.37.

2.5 Classical field theory

This section is a mere comment. In classical mechanics, the dynamical variables are the positions of particles, together with their velocities or momenta. For each particle there are three degrees of freedom of position and momenta.

In a field theory, such as Maxwell's theory, positions (and momenta) are not dynamical variables. After all, unlike a particle that moves around, a field can be everywhere at the same time. In the case of a field theory, what is dynamical is its amplitude.

Consider say a scalar field ϕ. At every point x in space it has an amplitude $\phi(x,t)$ that changes in time with a 'velocity' of $\dot{\phi}(x,t)$ which we may call the canonically conjugate momentum field: $\pi(x,t) := \dot{\phi}(x,t)$. Unlike the three degrees of freedom that particle possesses, a field therefore possesses uncountably many degrees of freedom, one at each position x. Now one can define the Poisson brackets of the first kind for them in analogy to the Poisson brackets for particles:

$$\{\phi(x,t), \pi(x',t)\} = \delta^3(x-x') \tag{2.39}$$
$$\{\phi(x,t), \phi(x',t)\} = 0 \tag{2.40}$$
$$\{\pi(x,t), \pi(x',t)\} = 0 \tag{2.41}$$

Here, $\delta^3(x-x')$ is the three dimensional Dirac delta distribution. The second set of axioms for the Poisson bracket is unchanged, i.e., it is still given by Eqs.2.9-2.13. There is also a representation of the Poisson bracket for fields in terms of derivatives but we won't cover it here (can you guess it?). The energy of the classical field, i.e., its Hamiltonian, is:

$$H(\phi,\pi) = \int d^3x\ \frac{1}{2}\left(\pi(x,t)^2 + \sum_{i=1}^{3}(\partial_i\phi(x,t))^2 + m^2\phi(x,t)^2\right) \tag{2.42}$$

The Hamilton equation Eq.2.38 is unchanged.

Exercise 2.14 *(Bonus question) Derive the equations of motion for $\phi(x,t)$ and $\pi(x,t)$, i.e., choose $f = \phi(x,t)$ or $f = \pi(x,t)$ in Eq.2.38. Combine these two differential equations by eliminating $\pi(x,t)$ to obtain one differential equation for $\phi(x,t)$ which contains up to the second time derivatives.*

The combined equation is the so-called Klein Gordon equation. The Dirac equation and the Maxwell equations can be treated similarly, although with some small extra complications because the amplitudes of these fields are not scalar but are vectorial and spinorial respectively.