QUANTIZING THE ELECTROMAGNETIC FIELD 1

Christopher Chamberland

Outline

- Summary of classical E&M
- Quantization of electromagnetic fields in free space (Coulomb gauge)
- Hamiltonian for quantized EM field
- Casimir effect
- Photon number eigenstates and coherent states
- Photon statistics

Summary of classical E&M (1)

- Coulomb gauge

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}; \quad \mathbf{B} = \nabla \times \mathbf{A} \qquad \nabla \cdot \mathbf{A} = 0, \quad \phi = 0$$

- In the vacuum, the vector potential obeys

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = 0$$

Solution for monochromatic field of definite polarization

$$\mathbf{A} = A_o \mathbf{1}_E (\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) + \exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t))$$

- E and B fields are found to be:

$$\mathbf{E} = i\omega A_o \mathbf{1}_E (\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) - \exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t))$$
$$\mathbf{B} = iA_o (\mathbf{k} \times \mathbf{1}_E)(\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) - \exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t))$$

Summary of classical E&M (2)

- Energy of the field obtained by integrating in space the energy density of the EM field

$$H_{\text{field}} = \int d^3r \left(\frac{\varepsilon_o E^2}{2} + \frac{B^2}{2\mu_o}\right)$$

- Integrate over volume V and use periodic B.C.

$$H_{\text{field}} = 2\varepsilon_o k^2 c^2 A_o^2 V$$

- Energy for n photons in the field

$$H_{\text{field}} = n\hbar\omega$$

- Allows us to choose constant A_0

$$A_o = \sqrt{\frac{\hbar}{2\varepsilon_o \omega V}}\, a$$

Define a to be $a \equiv \sqrt{n}$

Quantization of vector potential

- We showed the classically

$$\mathbf{A} = \sqrt{\frac{\hbar}{2\varepsilon_o \omega V}}\,\mathbf{1}_E[a\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) + a\exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t)]$$

with $a = \sqrt{n}$

- Quantize field by treating a as an operator $\hat{a}$
- Position demoted to status of parameter (like time)
- Generalizing to two independent polarization modes

$$\hat{\mathbf{A}} = \sum_{k,p}\sqrt{\frac{\hbar}{2\varepsilon_o \omega_k V}}\,\mathbf{1}_{Ep}[\hat{a}_{kp}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega_k t) + \hat{a}^+_{kp}\exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega_k t)]$$

Ensures A Hermitian

Quantized EM fields

- The electric and magnetic fields are

$$\hat{\mathbf{E}} = -\frac{\partial \hat{\mathbf{A}}}{\partial t} = i\sum_{k,p}\sqrt{\frac{\hbar}{2\varepsilon_o \omega_k V}}\mathbf{1}_{Ep}\,\omega_k [\hat{a}_{kp}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega_k t) - \hat{a}^{+}_{kp}\exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega_k t)]$$

$$\hat{\mathbf{B}} = \nabla\times\hat{\mathbf{A}} = i\sum_{k,p}\sqrt{\frac{\hbar}{2\varepsilon_o \omega_k V}}(\mathbf{k}\times\mathbf{1}_{Ep})[\hat{a}_{kp}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega_k t) - \hat{a}^{+}_{kp}\exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega_k t)]$$

$$[E_j(\vec{r},t), H_j(\vec{r}\,',t)] = 0 \quad (j = x, y, z)$$

$$[E_j(\vec{r},t), H_k(\vec{r}\,',t)] = -i\hbar c^2 \frac{\partial}{\partial l}\delta^{(3)}(\vec{r} - \vec{r}\,')$$

Cannot measure perpendicular components of E and B field simultaneously

- Quantum Hamiltonian obtained from

$$\hat{H}_{\text{field}} = \int d^3r[\frac{\varepsilon_o}{2}\hat{\mathbf{E}}^2 + \frac{1}{2\mu_o}\hat{\mathbf{B}}^2]$$

- Where we use

$$\int d^3r\exp(i\mathbf{k}\cdot\mathbf{r}) = V\delta_{\mathbf{k},0} \qquad \mathbf{1}_{Ep}\cdot\mathbf{1}_{Ep'} = \delta_{pp'}$$

Hamiltonian of EM field

- The Hamiltonian is then

$$\hat{H}_{\text{field}} = \frac{1}{2}\sum_{kp} \hbar\omega_k (\hat{a}_{kp}\hat{a}^{+}_{kp} + \hat{a}^{+}_{kp}\hat{a}_{kp})$$

$$[\hat{a}_{kp}, \hat{a}_{k'p'}{}^{+}] = \delta_{kk'}\delta_{pp'}; \quad [\hat{a}_{kp}, \hat{a}_{k'p'}] = [\hat{a}_{kp}{}^{+}, \hat{a}_{k'p'}{}^{+}] = 0$$

Quantum harmonic oscillator H

$$\hat{H} = \hbar\omega(\hat{a}^{+}\hat{a} + 1/2) = \hbar\omega(\hat{n} + 1/2)$$

$$[\hat{a}, \hat{a}^{+}] = 1$$

- Using the harmonic oscillator Lie algebra, we get

$$\hat{H}_{\text{field}} = \sum_{kp} \hbar\omega_k (\hat{a}^{+}_{kp}\hat{a}_{kp} + \frac{1}{2})$$

- Interpret $\hat{a}^{+}_{kp}\hat{a}_{kp} = \hat{n}_{kp}$ number operator for photons of frequency ω_k and polarization p

States of radiation field

- The states of the unperturbed radiation field are

$$|\ldots n_{kp} \ldots n_{k'p'} \ldots >= \ldots | n_{kp} > \ldots | n_{k'p'} > \ldots$$

- With

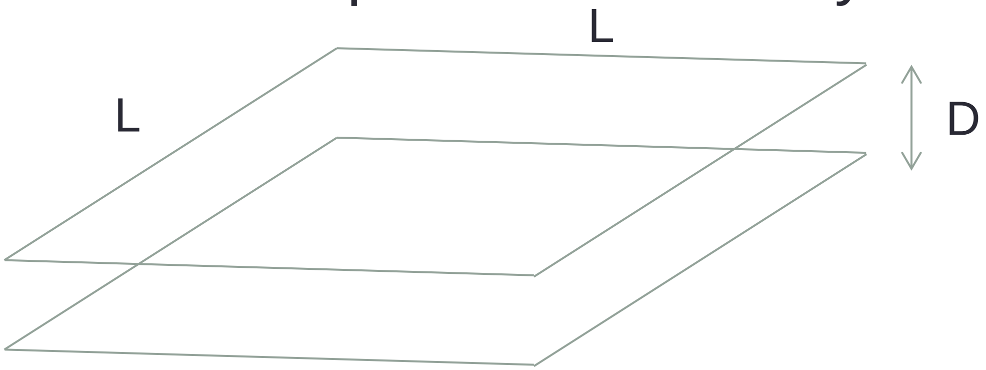
$$\hat{a}^{+}_{kp} |\ldots n_{kp} \ldots >= \sqrt{n_{kp}+1}\, |\ldots n_{kp}+1 \ldots >$$

$$\hat{a}_{kp} |\ldots n_{kp} \ldots >= \sqrt{n_{kp}}\, |\ldots n_{kp}-1 \ldots > .$$

- Divergent term $\sum_{kp} \hbar\omega_k / 2$ plays no role in radiative transitions.
- However plays important role in Casimir effect for the vacuum energy.

Casimir effect (1)

- Hendrik B.G. Casimir (1947)
- Boxed shape resonant cavity



- separated by distance D in z direction
- Walls made of conducting material. Walls in x-y plane have large size L
- Components of E field parallel to the wall must vanish at the wall.

Casimir effect (2)

- Maxwell's equations inside cavity:

$$\nabla^2 \vec{E}_{ox} = -\left(\frac{\omega}{c}\right)^2 E_{ox} \qquad E_{ox} = E_{ox}(x,y,z)$$

$$\nabla^2 \vec{E}_{oy} = -\left(\frac{\omega}{c}\right)^2 E_{oy} \quad \Rightarrow \quad E_{oy} = E_{oy}(x,y,z)$$

$$\nabla^2 \vec{E}_{oz} = -\left(\frac{\omega}{c}\right)^2 E_{oz} \qquad E_{oz} = E_{oz}(x,y,z)$$

- Solutions:

Integers n_x, n_y, n_z

$$E_x = E_o \cos(n_x \pi x / L) \sin(n_y \pi y / L) \sin(n_z \pi z / D) \cos(\omega_n t)$$
$$E_y = E_o \sin(n_x \pi x / L) \cos(n_y \pi y / L) \sin(n_z \pi z / D) \cos(\omega_n t)$$
$$E_z = E_o \sin(n_x \pi x / L) \sin(n_y \pi y / L) \cos(n_z \pi z / D) \cos(\omega_n t)$$

$$\omega_n = kc = c[(n_z \pi / D)^2 + \kappa^2]^{1/2}$$
$$\kappa^2 \equiv k_x^2 + k_y^2 = (n_x \pi / L)^2 + (n_y \pi / L)^2$$

Casimir effect (3)

- All modes with $n_i > 0$ have two independent polarization states.
- Modes for which one $n_i = 0$ have only one polarization state.
- Total vacuum energy

$$E(D) = \frac{1}{2}\sum_{n_i} \hbar\omega_n w_n \qquad w_n = \begin{cases} 2\ when\ all\ n_i\ positive \\ 1\ when\ only\ one\ n_i = 0 \\ 0\ otherwise \end{cases}$$

- L large, replace sum over n_x and n_y by integral over k

Casimir effect (4)

- Result

$$E(D) = \hbar \int (\sum_{n_z=1} \omega_n + \frac{1}{2} c\kappa) \frac{L^2 dk_x dk_y}{(2\pi)^2} = \frac{L^2 \hbar c}{2\pi} \int_0^\infty (\sum_{n_z=1} \sqrt{\kappa^2 + \pi^2 n_z^2 / D^2} + \frac{1}{2}\kappa) \kappa d\kappa$$

- Vacuum energy in absence of box

$$E_o = \hbar c \sum_{n_i} k \cong \frac{\hbar c L^2 D}{(2\pi)^3} \int_0^\infty k 4\pi k^2 dk$$

- Vacuum potential energy

$$U = E(D) - E_o$$

Dealing with divergent terms (1)

- Same divergent terms cancel out in subtraction
- Introduce cut-off k_c.
- Multiply integrand by $F(k) = \begin{cases} 1\ ;\ k \ll k_c \\ 0\ ;\ k \gg k_c \end{cases}$
- Result should not depend on cut-off for large k_c
- Take $F(k)=\exp(-k/k_c)$

$$U = \frac{L^2\hbar c}{2\pi}[\int_0^\infty (\sum_{n_z=1} \sqrt{\kappa^2 + \pi^2 n_z^2 / D^2} F(k) + \frac{1}{2}\kappa F(\kappa))\kappa d\kappa - \frac{D}{\pi}\int_0^\infty F(k) k^3 dk$$

$$= \frac{L^2\hbar c}{2\pi}(\sum_{n_z=1} \int_{n_z\pi/D}^\infty k^2 \exp(-k/k_c) dk + \int_0^\infty \frac{1}{2}\kappa^2 \exp(-\kappa/k_c) d\kappa - \frac{D}{\pi}\int_0^\infty \exp(-k/k_c) k^3 dk)$$

Dealing with divergent terms (2)

- Result $U = -\frac{L^2 \hbar c}{720} \frac{\pi^2}{D^3} + O(1/k_c^2)$

- Vacuum energy responsible for attracting pressure on walls separated by distance D.

$$P = -\frac{1}{L^2} \frac{dU}{dD} = -\frac{\hbar c}{240} \frac{\pi^2}{D^4}$$

- Experimentally observed 1997 by Steve Lamoreaux, Umar Mohideen and Anushree Roy.

Photon number eigenstates (1)

- We already saw that eigenstates of H are

$$|\ldots n_{kp} \ldots n_{k'p'} \ldots > = \ldots | n_{kp} > \ldots | n_{k'p'} > \ldots$$

- Corresponds to states with definite photon number for each mode k and polarization p.
- Recall E field

$$\vec{E}(\vec{r},t) = \sum_k \hat{\epsilon}_k \left[\varepsilon_k a_k e^{-i\nu_k t + ik\cdot r} + \varepsilon_k^* a_k^\dagger e^{i\nu_k t - ik\cdot r} \right] \qquad \varepsilon_k = \left(\frac{\hbar \nu_k}{2\epsilon_0 V} \right)^{1/2}$$

- Electric and magnetic field indefinite and fluctuating.

$$\left\langle \vec{E}(\vec{r},t) \right\rangle = 0$$

Probability distribution of E and B fields analogous to position and momentum distribution in energy eigenstate.

Photon number eigenstates (2)

- Second moment non-zero

$$\left\langle \left| \vec{E}\left(\vec{r},t\right) \right|^2 \right\rangle = \sum_k \left|\varepsilon_k\right|^2 \left(2n_k + 1\right)$$

- Divergence issue

- Circumvented by realising that particular experiment only couples to EM fields with finite bandwidth.

- Set cut-off's to modes resulting in finite sum.

Coherent states (1)

- Eigenstates of annihilation operators. $a_k \left|\alpha_k\right\rangle = \alpha_k \left|\alpha_k\right\rangle$
- For multiple modes can write

$$\left|\vec{\alpha}\right\rangle = \left|\alpha_1 \alpha_2 \ldots\right\rangle = \left|\alpha_1\right\rangle \otimes \left|\alpha_2\right\rangle \otimes \ldots$$

- Expectation value of E field same as solution to classical Maxwell equation

$$\left\langle \vec{E}\left(\vec{r}, t\right)\right\rangle = \sum_k \hat{\epsilon}_k \left[\varepsilon_k \alpha_k e^{-i\nu_k t + ik\cdot r} + \varepsilon_k^* \alpha_k^* e^{i\nu_k t - ik\cdot r}\right]$$

- α_k represents amplitude of classical field mode

- Coherent state gives closest quantum operator to classical field (but not equivalent to classical field).

Coherent states (2)

- Most light sources emit states of EM field close to coherent state (or statistical mixture of coherent states).

- Fluctuations independent of α_k

$$\left\langle \left| \vec{E}(\vec{r},t) \right|^2 \right\rangle - \left\langle \vec{E}(\vec{r},t) \right\rangle^2 = \sum_k |\varepsilon_k|^2$$

- Equal to mean square fluctuations in ground state

- Heisenberg inequality saturated when field is in coherent state.

Photon statistics

- Same photon number distribution in coherent state as for harmonic oscillator

$$P_\alpha(n_m) = \frac{|\alpha_m|^{2n}}{n!} e^{-|\alpha_m|^2}$$

$$|\alpha_k\rangle = e^{-\frac{|\alpha_k|^2}{2}} \sum_n \frac{(\alpha_k)^n}{\sqrt{n!}} |n\rangle$$

Coherent state expansion

- Poisson distribution with parameter $|\alpha_m|^2$ so that $\langle n_m \rangle = |\alpha_m|^2$

$$P(n) = \frac{\langle n\rangle^n}{n!} e^{-\langle n\rangle}$$

- Poisson distribution implies $\Delta n^2 = \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle$

- Fluctuations go to zero with many photons

$$\frac{\Delta n}{\langle n \rangle} = 1/\sqrt{\langle n \rangle}$$

Thank you!