Open quantum dynamics, Markov processes
Part I

QIC 895: Theory of Quantum Optics

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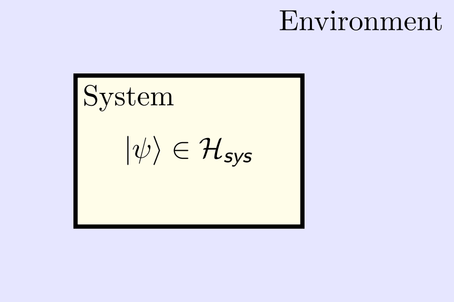
What is an 'open' system?

In contrast to a **closed system**



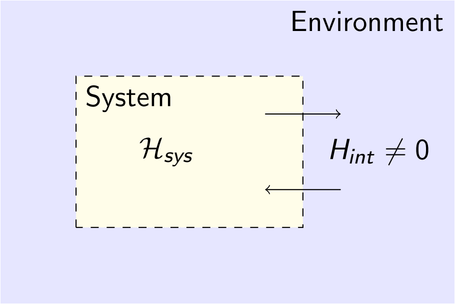
- No energy/information flow across boundary (i.e. $H_{int} = 0$)
- Described by $|\psi\rangle \in \mathcal{H}_{sys}$.

Evolution is unitary, governed by your favourite equation:

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle; \quad \partial_t\rho = \frac{1}{i\hbar}[H,\rho]; \quad U(x,t;x',t') = \int_{x'(t')}^{x(t)} \mathcal{D}q \, e^{\frac{i}{\hbar}S[q,\dot{q}]}$$

Open systems

Allow for energy flow across boundary



- Why would this be interesting?
- No system is an island. We want a more realistic model.
- Example: Quantum Control
  - Model noise
  - Model backreaction of measurements
  - Model effect of an actuator

Open systems

System evolution is generically NOT unitary!

- Interactions are typically entangling

$$\Longrightarrow \quad \rho_{sys} \text{ becomes mixed}$$
$$\Longrightarrow \quad \text{tr}\left[\rho_{sys}(T)^2\right] \leq \text{tr}\left[\rho_{sys}(0)^2\right]$$

- But unitaries preserve purity

$$\text{tr}\left[(U\rho_{sys}U^\dagger)^2\right] = \text{tr}\left[\rho_{sys}^2\right]$$

- Interactions will also change the system's entropy:

$$S(\rho_{sys}) = -\text{tr}(\rho_{sys}\log\rho_{sys}) \tag{1}$$

Problems to address

- How can we model the evolution of a subsystem of a larger, closed system?
- How general are these models?
- Are there models we can use to characterise typical behaviour?

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Operator sum

- Assume $\rho_{sys} \otimes |0\rangle\langle 0|$ on $\mathcal{H}_{sys} \otimes \mathcal{H}_{env}$
- Assume evolution is unitary

$$\rho_{sys} \otimes |0\rangle\langle 0| \mapsto U(\rho_{sys} \otimes |0\rangle\langle 0|)U^\dagger \tag{2}$$

- Trace out environment using ONB $\{|k\rangle\}_k$ for $\mathcal{H}_{env}$

$$\begin{aligned}
\Phi(\rho_{sys}) &:= \mathrm{tr}_{env}\left[U(\rho_{sys} \otimes |0\rangle\langle 0|)U^\dagger\right] \\
&= \sum_k \langle k| U(\rho_{sys} \otimes |0\rangle\langle 0|)U^\dagger |k\rangle \\
&= \sum_k E_k \rho_{sys} E_k^\dagger
\end{aligned} \tag{3}$$

where $E_k := \langle k| U |0\rangle$.

Operator sum

- Operator sum representation

$$\Phi(\rho_{sys}) = \sum_k E_k \rho_{sys} E_k^\dagger \tag{4}$$

- Φ characterised by operators $\{E_k\}_k$
- Completeness relation: $\sum_k E_k^\dagger E_k = \mathbb{1}$

$$\begin{aligned} \mathrm{tr}[\Phi(\rho_{sys})] &= \mathrm{tr}[\sum_k E_k \rho_{sys} E_k^\dagger] \\ &= \mathrm{tr}[\sum_k E_k^\dagger E_k \rho_{sys}] \\ &= \mathrm{tr}[\rho_{sys}] = 1 \end{aligned} \tag{5}$$

- Indeed, this will be the case for $E_k = \langle k| U |0\rangle$

Operator sum

e.g. (Measurement)

- $\mathcal{H}_{sys} \otimes \mathcal{H}_{env} = \mathbb{C}^2 \otimes \mathbb{C}^2$
- Find $\Phi(\rho_{sys})$ for ρ_{sys} and $\rho_{env} = |0\rangle\langle 0|$, with

$$U = P_0 \otimes \mathbb{1} + P_1 \otimes \sigma_x \tag{6}$$

where $P_0 := |0\rangle\langle 0|$, $P_1 := |1\rangle\langle 1|$, $\sigma_x := |0\rangle\langle 1| + |1\rangle\langle 0|$.
Solution:

$$\begin{aligned} E_0 &:= \langle 0| U |0\rangle = P_0 \langle 0| \mathbb{1} |0\rangle + P_1 \langle 0| \sigma_x |0\rangle = P_0 &(7)\\ E_1 &:= \langle 1| U |0\rangle = P_0 \langle 1| \mathbb{1} |0\rangle + P_1 \langle 1| \sigma_x |0\rangle = P_1 &(8)\\ & &(9) \end{aligned}$$

- Operator sum:

$$\Phi(\rho_{sys}) = P_0 \rho P_0 + P_1 \rho P_1 \tag{10}$$

Converse: Extending operator sum

Problem: the environment is a *big* place.
What if we don't have a complete model for its dynamics?

- Suppose we start with $\{E_k\}_k$ s.t. $\sum_k E_k^\dagger E_k = \mathbb{1}$
- Can we find suitable $\mathcal{H}_{env}$ so that evolution on $\mathcal{H}_{sys} \otimes \mathcal{H}_{env}$ is unitary?

Yes!

- Let $\mathcal{H}_{env}$ with ONB $\{|k\rangle\}_k$ (i.e. one $|k\rangle$ for each E_k)
- Define U on $\mathcal{H}_{sys} \otimes \mathcal{H}_{env}$ by

$$U|\psi\rangle|0\rangle := \sum_k E_k|\psi\rangle|k\rangle \tag{11}$$

Unitary:

$$\langle\phi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \sum_k \langle\phi|E_k^\dagger E_k|\psi\rangle = \langle\phi|\psi\rangle \tag{12}$$

(can extend to a unitary on all $\mathcal{H}_{sys} \otimes \mathcal{H}_{env}$)
Evolution:

$$\mathrm{tr}_{env}[U(\rho_{sys} \otimes |0\rangle\langle 0|)U^\dagger] = \sum_k E_k \rho_{sys} E_k^\dagger \tag{13}$$

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Amplitude damping

Amplitude damping = Loss of energy

e.g. (Spontaneous emission)

- $\mathcal{H}_{sys} = \mathcal{H}_{atom}$ and $\mathcal{H}_{env} = \mathcal{H}_{EM}$
- $\rho_{env} = |0\rangle\langle 0|$
- Model for (unitary) evolution of $\mathcal{H}_{sys} \otimes \mathcal{H}_{env}$:

$$\begin{aligned} |g\rangle|0\rangle &\mapsto |g\rangle|0\rangle \\ |e\rangle|0\rangle &\mapsto \sqrt{1-p}\,|e\rangle|0\rangle + \sqrt{p}\,|g\rangle|1\rangle \end{aligned}$$

- Operator elements:

$$E_0 = \langle 0|U|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$$
$$E_1 = \langle 1|U|0\rangle = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \tag{14}$$

Amplitude damping

$$\Phi(\rho_{sys}) = E_0 \rho_{sys} E_0^\dagger + E_1 \rho_{sys} E_1^\dagger \tag{15}$$

Write

$$\rho_{sys} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \tag{16}$$

then

$$\Phi(\rho_{sys}) = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix} \tag{17}$$

Amplitude damping

- Suppose we decided to apply Φ n times:

$$\begin{aligned}\Phi^{(n)}(\rho_{sys}) &:= \Phi(\Phi(\ldots\Phi(\rho_{sys})\ldots)) \\ &= \begin{pmatrix} \rho_{00} + p^n(1-p)^{n-1}\rho_{11} & (1-p)^{n/2}\rho_{01} \\ (1-p)^{n/2}\rho_{10} & (1-p)^n\rho_{11} \end{pmatrix}\end{aligned} \tag{18}$$

Suppose p is prob of decaying for interval δt, say $p = \Gamma\delta t$.
Total time $t = n\delta t$, then $p = \Gamma\delta t = \frac{\Gamma t}{n}$.
Recall: $\lim_{n\to\infty}(1 - \frac{\Gamma t}{n})^n = e^{-\Gamma t}$
Thus,

$$\Phi^{(\infty)}(\rho_{sys}) = \begin{pmatrix} \rho_{00} + (1-e^{-\Gamma t})\rho_{11} & e^{-\Gamma t/2}\rho_{01} \\ e^{-\Gamma t/2}\rho_{10} & e^{-\Gamma t}\rho_{11} \end{pmatrix} \tag{19}$$

We see that probability that atom stays in excited state decays as $e^{-\Gamma t}$, as found previously.

Phase damping

Phase damping = Loss of information *without* loss of energy

- Happens when system interacts weakly with many subsystems in the environment.
- Will only change relative phase between energy eigenstates.

e.g. Model for evolution on $\mathcal{H}_{sys} \otimes \mathcal{H}_{env}$:

$$\begin{aligned} |g\rangle|0\rangle &\mapsto \sqrt{1-p}\,|g\rangle|0\rangle + \sqrt{p}\,|g\rangle|1\rangle \\ |e\rangle|0\rangle &\mapsto \sqrt{1-p}\,|e\rangle|0\rangle + \sqrt{p}\,|e\rangle|2\rangle \end{aligned} \tag{20}$$

then,

$$\begin{aligned} E_0 = \langle 0|\,U\,|0\rangle &= \sqrt{1-p}\mathbb{1} \\ E_1 = \langle 1|\,U\,|0\rangle &= \sqrt{p}P_g \\ E_2 = \langle 2|\,U\,|0\rangle &= \sqrt{p}P_e \end{aligned}$$

Phase damping

$$\begin{aligned}\Phi(\rho_{sys}) &= (1-p)\rho_{sys} + pP_g\rho_{sys}P_g + pP_e\rho_{sys}P_e \\ &= \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}\end{aligned} \tag{21}$$

If we go through procedure as before where $p = \Gamma\delta t$, $t = n\delta t$,

$$\Phi^{(n)}(\rho_{sys}) = \begin{pmatrix} \rho_{00} & (1-p)^n\rho_{01} \\ (1-p)^n\rho_{10} & \rho_{11} \end{pmatrix} \tag{22}$$

then,

$$\Phi^{(\infty)}(\rho_{sys}) = \begin{pmatrix} \rho_{00} & e^{-\Gamma t}\rho_{01} \\ e^{-\Gamma t}\rho_{10} & \rho_{11} \end{pmatrix} \tag{23}$$

- We see that for $t \to \infty$, $\Phi^{(\infty)}(\rho_{sys})$ is an incoherent superposition of eigenstates.
- We also see that $\mathrm{tr}[H_{free}\rho_{sys}] = \mathrm{tr}[H_{free}\Phi^{(n)}(\rho_{sys})]$.

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Superoperators

Step back:

- Suppose we wanted to write down a general $\Phi : \rho \mapsto \rho'$.
- What properties should Φ have?

1. Hermiticity preserving: $\Phi(\rho)^\dagger = \Phi(\rho)$ if $\rho^\dagger = \rho$
2. Trace preserving: $0 \leq \mathrm{tr}[\Phi(\rho)] \leq 1$
3. Linear: $\Phi\left(\sum_i p_i \rho_i\right) = \sum_i p_i \Phi(\rho_i)$
4. Positive: $\Phi(\rho) \geq 0$ for any $\rho \geq 0$
5. Completely positive: $(\mathbb{1} \otimes \Phi)(\rho) \geq 0$ for any $\rho \geq 0$

Completely positive maps

Why completely positive?

e.g. Transpose

- Single system $\mathcal{H}$, operator ρ

$$\Phi : \rho = \sum_i p_i \left|p_i\right\rangle \left\langle p_i\right| \mapsto \rho^T = \rho \tag{24}$$

- Two qubit system
  - $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathcal{H}_{sys} \otimes \mathcal{H}_{env}$
  - $\Phi \otimes \mathbb{1}$ transposes first qubit, does nothing to second

$$\begin{aligned}(\Phi \otimes \mathbb{1}) :& \tfrac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \\ &\mapsto \tfrac{1}{2}(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|)\end{aligned} \tag{25}$$

  - Eigenvalues of $(\Phi \otimes \mathbb{1})(|\psi\rangle\langle\psi|)$ are: $+\frac{1}{2}$, $+\frac{1}{2}$, $+\frac{1}{2}$, and $-\frac{1}{2}$.

Kraus Representation theorem

thm. The map Φ is a superoperator (i.e. satisfies the properties 1-5 above), *iff*

$$\Phi(\rho) = \sum_k E_k \rho E_k^\dagger \tag{26}$$

for some set of operators $\{E_k\}_k$ such that $\sum_k E_k^\dagger E_k \leq \mathbb{1}$.

Next time:

Master equations

- Describe continuous time evolution of quantum channels:

$$\dot{\rho}(t) = -\, i\,[H, \rho(t)] + \sum_k \left(L_k \rho(t) L_k^\dagger - \tfrac{1}{2} L_k^\dagger L_k \rho(t) - \tfrac{1}{2} \rho(t) L_k^\dagger L_k \right) \tag{27}$$

References

1. P. Cappellaro, *Quantum Theory of Radiation Interactions*, MIT Open Courseware, Course 22.51 (2012).
2. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2000).

Questions?