Note Title

Project: Simulate quantum field fluctuations

Task:
- For various important states of scalar QFT, calculate the probability distribution for finding certain outcomes when 'measuring' all $\hat{\phi}(x)$ simultaneously.
- Then draw from these probability distributions and plot the results.
- Discuss your findings.

Project structure:
- Parts A + B as described below.
- To get you started, I explain A in more detail.
- Set your own emphasis. No need to cover each point.

A) On Minkowski space

Consider the Klein Gordon equation

$$(\partial_t^2 - \Delta + m^2)\,\phi(x,t) = 0$$

in a box $[0,L]\times[0,L]\times[0,L]$ with Dirichlet boundary conditions:

$$\phi(\text{boundary}) = 0$$

Recall that if the box is chosen large enough, the physics of the boundaries does not matter in the middle.

But the above boundary conditions have the mathematical advantage that we can use the discrete Fourier sine transform:

Discrete Fourier sine transform:

The square integrable, twice differentiable functions on an interval $[0, L]$, which vanish at the boundaries are spanning a Hilbert space $\mathcal{F}$.

An ON basis of $\mathcal{F}$ is given by the set of functions
[a "Hilbert basis"]

$$b_n(x) := \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right), \text{ where } n = 1, 2, \ldots$$

i.e., we have $\int_0^L b_n(x)\, b_m(x)\, dx = \delta_{nm}$ and therefore for any $f \in \mathcal{F}$:

$$f(x) = \sum_{n=1}^{\infty} f_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \text{ with } f_n := \int_0^L f(x) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) dx.$$

Tasks:

* Use this transform to obtain a mode decomposition of $\phi(x,t)$ with coefficients $\phi_n(t)$.

  $n = (n_1, n_2, n_3)$

* Quantize by translating the equations

$$\left(\partial_t^2 - \Delta + m^2\right) \hat{\phi}(x,t) = 0$$

$$[\hat{\phi}(x,t), \dot{\hat{\phi}}(x',t)] = i\, \delta(x - x') \qquad \text{etc}$$

$$\hat{\phi}(x,t)^+ = \hat{\phi}(x,t)$$

  into equations that the $\hat{\phi}_n(t)$ must obey.

* You should arrive at harmonic oscillators. Assume they are in their joint ground state, which is the vacuum state. Calculate the probability distribution for each $\hat{\phi}_n(t)$.

* Draw ϕ_n measurement outcomes from those distributions and plot the resulting $\phi(x)$.
  - ☐ In practice, you can only use finitely many coefficients ϕ_n. How does the resulting picture change as you take more and more coefficients into account? What do you expect in the limit of all ϕ_n taken into account?
  - ☐ What effect does the mass m have?
  - ☐ Plot a case when space has 2 dimensions and compare with 2-dim slices of cases of space having more dimensions. (In each case, keep the length L in any direction the same.)
  - ☐ Draw and plot the case of a wave packet state.

B) Inflating FRW spacetimes.

(Using conformal time and comoving coordinates)

We then define $\hat{\chi}(x,t) = a(t)\,\hat{\phi}(x,t)$ (see lecture)

with $\hat{\chi}''(x,t) - \Delta\hat{\chi}(x,t) + \left(m^2a^2 - \frac{a''}{a}\right)\hat{\chi}(x,t) = 0.$

Tasks:

- ☐ Perform a Fourier sine mode decomposition to obtain the $\hat{\chi}_n$ and the equations they obey. Interpret the box: Does it stay the same "size"?

- ☐ Expand as usual: $\hat{\chi}_n = v_n a_n + v_n^* a_n^+$. Notice that the same mode functions that we used so far (Hankel or Bessel functions) can again be used for the $v_n(\tau)$.

- We need to find the probability distribution for measurement outcomes if we measured $\hat{x}_n$.

So we need to calculate the eigenstates of $\hat{x}_n$,

$$\hat{x}_n |\lambda\rangle = \lambda |\lambda\rangle$$

Then if $|\psi\rangle$ is the state of the QFT (usually $|\psi\rangle = |0\rangle$) we have:

probability for finding λ when measuring $\hat{x}_n$ is:

$$p(\lambda) = |\langle\lambda|\psi\rangle|^2 \qquad (S)$$

- To this end, recall from QM:

For $\hat{x} := \frac{1}{\sqrt{2\mu}}(a^+ + a)$

the position eigenvectors in the Fock basis are:

$$\hat{x}|x\rangle = x|x\rangle, \qquad |x\rangle = \sum_r |r\rangle\langle r|x\rangle$$

Fock basis

with:

$$\boxed{\langle x|r\rangle = \frac{1}{\sqrt{2^r r!}}\left(\frac{\mu}{\pi}\right)^{1/4} H_r(\sqrt{\mu}\,x)\, e^{-\frac{\mu x^2}{2}} \qquad (*)}$$

- Here, we have $\hat{x}_n = \hat{x}$ and $\frac{1}{\sqrt{2\mu}} = |v(\eta)|$

(the phase of $v(\eta)$ drops out).

$\Rightarrow$ Eq. $(*)$ yields distribution (S').

- Choose an η, draw from prob. distr. for $|\psi\rangle = |0\rangle$, then plot.

Format of project report:

- 15 - 20 pages
- Title / Abstract / Introduction /
  Motivation / Theory / Method / Results / Discussion
  ⋮ repeated for each sub project
  Motivation / Theory / Method / Results / Discussion
- Conclusions / Suggestions for further study
- Bibliography and software used
- No need to stick to exactly that format.

Recall: Descriptions are fine but explanations are what we are after.