Quantum theory of damping
Heisenberg-Langevin approach

QIC 895: Theory of Quantum Optics

Marian Berek
Institut for Quantum Computing, UW
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How we can model the system-reservoir interaction ?

- Quantum operator approach
- Suitable for the calculation of two-time correlation functions of the system operator
- How dissipation by the system is related to the fluctuation of the reservoir ?
- What is the difference between spontaneous decay rate of atom in free space and cavity ?

Outline

- 4 models
  - damping of a single mode field
    - oscillator reservoir
    - atomic reservoir
  - multi oscillator heat bath problem
  - atom in damped cavity
- Fluctuation dissipation theorem
- The spontaneous decay rate of atom
  - free space vs inside the cavity
- Field correlation function

General steps in the approach

- $H=H_S+H_R+H_I=H_0+H_I$
  - justify assumption of the model
  - use approximation if possible
- Derive equation for a(t) and other interesting operators
  - find Langevin noise operator
- Calculate the correlation functions

Meaning of the correlation functions

- correlation of the operators at two time points
- calculate for example field spectrum

$$\langle F_a^\dagger(t) F_a(t') \rangle$$
$$\langle F_a(t) F_a^\dagger(t') \rangle$$
$$\langle a_a(t) F_a^\dagger(t') \rangle$$
$$\langle F_a^\dagger(t) a_a(t') \rangle$$

Models

	System	**Reservoir**
Markovian white noise	single mode field	many oscillators
non Markovian color noise	single mode field	ensemble of atoms
non Markovian color noise	oscillator	many oscillators
vacuum modes that enter cavity through partially transparent mirrors	atom +single mode in cavity	many oscillators

Markovian white noise
single mode field + many oscillators
equation for noise operator

$$H_S = \hbar \nu a^\dagger a$$

$$H_R = \hbar \sum_k \nu_k b_k^\dagger b_k$$

$$H_I = \hbar \sum_k g_k \left(b_k^\dagger a_k + a_k^\dagger b_k \right)$$

$$\dot{a} = \frac{i}{\hbar}[H, a] = -i\nu a(t) - i \sum_k g_k b_k(t)$$

$$\dot{b}_k = -i\nu_k b_k(t) - i g_k a(t)$$

$$\dot{a} = -i\nu a - \sum_k g_k^2 \int_0^t dt' a(t') e^{(-i\nu_k(t-t'))} + f_a(t)$$

$$noise\, operator:$$

$$f_a(t) = -i \sum_k g_k b_k(0) e^{(-i\nu_k t)}$$

Markovian white noise
single mode field + many oscillators

$$\tilde{a}(t) = a(t)e^{i\nu t}$$

$$\dot{\tilde{a}}(t) = -\sum_k g_k^2 \int_0^t dt'\,\tilde{a}(t')\,e^{(-i(\nu_k - \nu)(t-t'))} + F_{\tilde{a}}(t)$$

the Langevin equation:

$$\dot{\tilde{a}} = -\frac{1}{2}\eta\tilde{a} + F_{\tilde{a}}(t)$$

damping constant: $\eta = 2\pi[g(\nu)]^2 D(\nu)$

$g(\nu)$ − *coupling constant evaluated at* $\nu = k/c$

$D(\nu)$ − *density of states*

$$F_{\tilde{a}}(t) = e^{(i\nu t)} f_a(t) = -i\,e^{(i\nu t)} \sum_k g_k b_k(0)\,e^{(-i\nu_k t)}$$

Markovian white noise
single mode field + many oscillators

$$if\ reservoir\ is\ \text{in}\ thermal\ equilibrium:$$

$$\langle b_k(0)\rangle_R = \langle b^{\dagger}_{\mathrm{k'}}(0)\rangle_R = 0$$

$$\langle b^{\dagger}_k(0)\,b_{\mathrm{k'}}(0)\rangle_R = \delta_{\mathrm{kk'}}\bar{n}_k$$

$$\langle b_k(0)\,b^{\dagger}_{\mathrm{k'}}(0)\rangle_R = (\bar{n}_k + 1)\,\delta_{\mathrm{kk'}}$$

$$\langle b_k(0)\,b_{\mathrm{k'}}(0)\rangle_R = \langle b^{\dagger}_k(0)\,b^{\dagger}_{\mathrm{k'}}(0)\rangle_R = 0$$

Markovian white noise
single mode field + many oscillators

$$\langle F_{\tilde{a}}(t)\rangle_R = \langle F_{\tilde{a}}^{\dagger}(t)\rangle_R = 0$$

$$\langle F_{\tilde{a}}^{\dagger}(t) F_{\tilde{a}}(t')\rangle_R = \int_0^{\infty} D(\nu_k)[g(\nu_k)]^2 \bar{n}(\nu_k)\exp(i(\nu k-\nu)(t-t'))\,d\nu_k$$

$$slowly\ varying\ terms: at\ \nu_k=\nu$$

$$D(\nu_k), g(\nu_k)\bar{n}(\nu_k)=\bar{n}_{\mathrm{th}}$$

$$\text{white noise}, delta\ correlation\ function:$$

$$\langle F_{\tilde{a}}^{\dagger}(t) F_{\tilde{a}}(t')\rangle_R = \eta\bar{n}_{\mathrm{th}}\delta(t-t')$$

$$diffusion\ coefficient:$$

$$2\langle D_{\tilde{a}^{\dagger}\tilde{a}}\rangle = \bar{n}_{\mathrm{th}}$$

non Markovian colored noise
single mode field + ensemble of atoms

$$H_0 = \hbar v a^\dagger a + \frac{1}{2}\sum_k \hbar v \sigma_z$$

$$H_I = \hbar g \sum_i \left[f\left(t, t_i, \tau\right) a_k^\dagger \sigma_-^i + H.c. \right]$$

atoms passing through a cavity:

− monoenergetic

− long lived

− resonant with the field

− interact during τ period

$$f\left(t_i, t, \tau\right) = 1 \; for \; t_i \le t < t_i + \tau$$

$$f\left(t_i, t, \tau\right) = Otherwise$$

non Markovian colored noise
single mode field + ensemble of atoms

$$\dot{a} = -\frac{1}{2}\eta a + F_a(t)$$

damping constant:

$$\eta = -2g^2 \sum_j \int_{t_1}^{t_2} dt' f(t_i, t, \tau) f(t_i, t', \tau) \sigma_z^i(t_i)$$

noise operator:

$$F_a(t) = -ig \sum_i f(t_i, t, \tau) \sigma_-^i(t_i)$$

colored noise:

$$\langle F_a^\dagger(t) F_a(t') \rangle = \alpha_F(\tau - |(t-t')| / \tau^2) \; for \; |(t-t')| \le \tau$$

$$\langle F_a^\dagger(t) F_a(t') \rangle = 0 \; otherwise$$

$$\alpha = r_a g^2 \tau^2 [1 + \exp(\hbar \nu / k_B T)]^{-1}$$

r_a − *the rate of injection of atoms into the cavity*

oscillator + many oscillators

$$H=\frac{p^2}{2m}+\frac{1}{2}m\nu^2x^2+\sum_j\left[\frac{p_j^2}{2m_j}+\frac{1}{2}m_j\nu_j^2(q_j-x)^2\right]$$

general expression for damped oscillator:

$$m\ddot{x}(t)+\int_{-\infty}^{t}dt'\mu(t-t')\dot{x}(t')+m\nu^2x=F(t)$$

damping function:

$$\mu(t-t')=\sum_j m_j\omega_j^2\cos[\omega_j(t-t')]$$

for constant damping colored noise:

$$\frac{1}{2}\langle F(t)F(t')+F(t')F(t)\rangle=\Gamma k_BT\frac{d}{dt}\coth[\pi k_BT(t-t')/\hbar]$$

$$F(t)=\sum_j m_j\omega_j^2q_j^0(t)$$

atom + single mode in cavity + thermal reservoir

$$H_F = \hbar \nu a^{\dagger} a$$

$$H_A = \frac{1}{2} \hbar \nu \sigma_z$$

$$H_R = \hbar \sum_k \nu_k b_k^{\dagger} b_k$$

vacuum modes that enter the cavity through partially transmitting mirrors

$$H_{FR} = \hbar \sum_k g_k \left(b_k^{\dagger} a_k + a_k^{\dagger} b_k \right)$$

$$H_{AR} = \hbar g_k \left(\sigma_+ a + a_k^{\dagger} \sigma_- \right)$$

$$\frac{d}{dt} \left[(a^{\dagger})^m a^n O_A \right] = -\frac{i}{\hbar} \left[(a^{\dagger})^m a^n O_A , H_F + H_A + H_R \right] + \left\langle \frac{d}{dt} \left[(a^{\dagger})^m a^n \right] \right\rangle_R O_A$$

$$\frac{d}{dt} \langle a^{\dagger} a \rangle = ig \langle \sigma_+ a - a^{\dagger} \sigma_- \rangle - \eta \langle a^{\dagger} a \rangle + \eta \bar{n}_{\mathrm{th}}$$

$$\frac{d}{dt} \sigma_z = -2 ig \langle \sigma_+ a - a^{\dagger} \sigma_- \rangle$$

atom + single mode in cavity + thermal reservoir

$simplify - set\ initial\ condition:$

$atom: |e\rangle$

$cavity\ field: |0\rangle, \bar{n}_{\mathrm{th}}$

$\langle (a^{\dagger})^2 \sigma_z a^2 \rangle = 0$

$the\ set\ of\ four\ equations\ can\ be\ solved$

$$\langle a^{\dagger} a \rangle_t = - \frac{8 g^2 e^{(-\eta t/2)}}{\eta^2 - 16 g^2} \{ 1 - \cosh[(\eta^2 - 16 g^2)^{1/2} t/2] \}$$

$$\langle \sigma_z \rangle_t = -1 + \frac{4 e^{(-\eta t/2)}}{\eta^2 - 16 g^2} \{ -4 g^2 + [\frac{\eta^2}{4} - 2 g^2 + \frac{\eta}{4} (\eta^2 - 16 g^2)^1 / 2] e^{((\eta^2 - 16 g^2)^{1/2} t/2)} + [\frac{\eta^2}{4} - 2 g^2 - \frac{\eta}{4} (\eta^2 - 16 g^2)^1 / 2] e^{(-(\eta^2 - 16 g^2)^{1/2} t/2)} \}$$

The spontaneous decay rate of atom free space vs inside the cavity

- the spontaneous decay rate in general

$$\Gamma = 2\pi \langle |(g(\omega))^2| \rangle_{angular} D(\omega)$$
$$g(\omega) - vacuum\, Rabi\, frequency$$
$$D(\omega) - density$$

- for a cavity tuned near the atomic resonance frequency

$$\Gamma_c = \Gamma_{fs} Q \left(\frac{2\pi c^3}{V \omega^3}\right)$$

Field correlation function

$$\frac{d}{dt}\langle \tilde{a} \rangle_R = -\frac{1}{2}\eta \langle \tilde{a} \rangle_R$$

$$\frac{d}{dt}\langle \tilde{a}(t)^\dagger \tilde{a}(t) \rangle_R = -\eta \langle \tilde{a}(t)^\dagger \tilde{a}(t) \rangle_R + \eta \bar{n}_{\mathrm{th}}$$

damping of the field inside the cavity at the rate $\eta = \nu / Q$

$$\langle \tilde{a}(t_i)^\dagger \tilde{a}(t_i + \tau) \rangle_R = \langle \tilde{a}(t_i)^\dagger \tilde{a}(t_i) \rangle_R \exp\left(-\frac{\nu}{2Q}\tau\right)$$

the field spectrum (Lorentzian distribution):

$$S(\omega) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty \langle a(t_i)^\dagger a(t_i + \tau) \rangle_R \exp(i\omega\tau) = \frac{\langle n \rangle}{\pi} \frac{\nu / 2Q}{(\omega - \nu)^2 + (\nu / 2Q)^2}$$

$$D_c(\omega) = \frac{S(\omega)}{\langle n \rangle}$$

Fluctuation - dissipation theorem

- second-order correlation function of the Langevin noise

$$\langle F^{\dagger}(t)_{\tilde{a}} F(t')_{\tilde{a}} \rangle_R = \eta n_{th} \delta(t - t')$$

- Calculate the system damping

$$\eta = \frac{1}{n_{th}} \int_{-\infty}^{\infty} \langle F^{\dagger}(t)_{\tilde{a}} F(t')_{\tilde{a}} \rangle_R \, dt'$$

Summary

- **Quantum Langevin** equation are useful for calculation different **correlation functions**

- **dissipation** is a **result** of **fluctuation** of the reservoir

- **spontaneous decay rate** of atom is **enhanced** in **cavity**

- **probability of the atom being in the upper level** strongly depends **damping** and **coupling constant**

References

- Quantum Optics, Scully & Zubairy (1997)