University of Waterloo
Department of Pure Mathematics

Analysis Comprehensive Exam 2021
Measure Theory and Fourier Analysis

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REMINDER: The Comprehensive Exams will be static open book, in the sense that consultation of all static pre-downloaded materials will be allowed. Examples of allowable aids include standard texts, and personal study notes. However, no other aids are permitted. In particular, discussing (over any medium) the exam with any person during the availability period, or using the internet or any other non-static tool to search questions or concepts, is not allowed.

Your submitted solutions must reflect your own understanding of the concepts being tested, in your own words.

When you submit a Comprehensive Exam, you are agreeing to the following Academic Integrity Statement:

INTEGRITY STATEMENT

I declare the following statements to be true:

1. The work I submit here is entirely my own.
2. I have not used any unauthorized aids.
3. I have not discussed and will not discuss the contents of this examination with anyone until after the submission deadline.
4. I am aware that misconduct related to examinations can result in significant penalties, including failing the examination and suspension (this is covered in Policy 71: `https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-71`)

Instructions: Answer all questions.

Let $C[a,b]$ denote the space of complex-valued continuous functions on a compact interval $[a,b]$, equipped with the uniform norm $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$. Let $L^p[a,b]$ denote the space of complex-valued Lebesgue p-integrable functions.

1. (a) Show that the simple functions are dense in $L^\infty[0,1]$.

   (b) Show that for every function f in $L^\infty[0,1]$, there is a sequence $(f_n)_{n=1}^\infty$ of continuous functions in $L^\infty[0,1]$ with the property that for every function $g \in L^1[0,1]$,
$$\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg.$$

   (c) Must the sequence $(f_n)_{n=1}^\infty$ from (b) be uniformly bounded?

2. For $f \in L^1[0,2\pi]$, define $F : [0,2\pi] \to \mathbb{C}$ by $F(x) = \displaystyle\int_0^x f$ for $x \in [0,2\pi]$.

   (a) Show that F is continuous.

   (b) Let $(f_n)_{n=1}^\infty$ be a sequence in $L^1[0,2\pi]$ satisfying $\lim\limits_{n\to\infty} \|f_n - f\|_1 = 0$. Define $F_n : [0,2\pi] \to \mathbb{C}$ by $F_n(x) = \int_0^x f_n$ for $x \in [0,2\pi]$. Show that the sequence $(F_n)_{n=1}^\infty$ converges uniformly to F.

   (c) Recall that the Fourier coefficients of a function $g \in L^1[0,2\pi]$ are
$$\widehat{g}(n) = \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-inx}\, dx, \quad n \in \mathbb{Z}.$$
   Suppose that f has mean value zero, i.e. $\widehat{f}(0) = 0$. Show that for all $n \in \mathbb{Z}$,
$$\widehat{F}(n) = \frac{1}{in} \hat{f}(n).$$

   (d) Show that if f is square integrable and has mean value-zero, then
$$F(x) = \sum_{n=-\infty}^{\infty} \widehat{F}(n) e^{inx}, \quad x \in [0,2\pi],$$
   where the convergence of the series is uniform.