On the number of prime factors of integers of the form $ab + 1$

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1 Introduction

For any X let $|X|$ denote its cardinality and for any integer n, larger than one, let $w(n)$ denote the number of distinct prime factors of n and let $P(n)$ denote the greatest prime factor of n. In 1934 Erdös and Turan [5] proved that there exists a positive number c_1 such that for any non-empty set A of positive integers

$$w\left(\prod_{a,a^1\epsilon A}(a+a^1)\right) > c_1 \log|A| \quad . \tag{1.1}$$

In 1986, György, Stewart and Tijdeman [12] proved that this result can be extended to the case when the summands are taken from different sets. They proved that there is a positive number c_2 such that for any sets A and B of positive integers with $|A| \geq |B| \geq 2$ we have

$$w\left(\prod_{a\epsilon A, bB}(a+b)\right) > c_2 \log|A| \quad . \tag{1.2}$$

Moreover, in 1988, Erdös, Stewart and Tijdeman [4] showed that (1.2) is not far from best possible. They proved that there is a positive number c_3 such that for each integer k, with $k \geq 2$, there exist sets of positive integers A and B with $k = |A| \geq |B| \geq 2$ satisfying

$$w\left(\prod_{aA,bB}(a+b)\right) < c_3(\log|A|)^2 \log\log\ |A| \quad . \tag{1.3}$$

If the sets A and B are dense sets of integers then estimates (1.1) and (1.2) may be strengthened. Let ϵ and δ be positive real numbers and let N

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be a positive integer. Let A and B be subsets of $\{1, \ldots, N\}$ of cardinality at least δN. In [3], Erdös, Pomerance, Sárközy and Stewart proved that there exists a positive number N_0, which is effectively computable in terms of ϵ and δ, such that if N exceeds N_0 then there exists an integer a from A and an integer b from B for which

$$w(a+b) > (1-\epsilon)(\log N)/\log\log\ N \quad . \tag{1.4}$$

Sárközy and Stewart [17] were able to show that a lower bound of the same order of magnitude holds even under a much weaker density condition. Let θ be a real number with $\frac{1}{2} < \theta \leq 1$ and let N be a positive integer. They proved that there exists a positive number c_4, which is effectively computable in terms of θ, such that if A and B are subsets of $\{1, \ldots, N\}, N$ exceeds c_4 and

$$(|A||B|)^{1/2} \geq N^{\theta} \quad ,$$

then there exists an integer a from A and an integer b from B for which

$$w(a+b) > \frac{1}{6}\left(\theta - \frac{1}{2}\right)^2 (\log N)/\log\log\ N \quad . \tag{1.5}$$

In the same article [17], they estimated the average value of $w(a+b)$. They showed that if A and B are subsets of $\{1, \ldots, N\}$ with $(|A||B|)^{1/2} = N\exp(-(\log N)^{0(1)})$ then

$$\frac{1}{|A|\ |B|}\sum_{a\epsilon A}\ \sum_{b\epsilon B} w(a+b) > (1+0(1))\log\log\ N \quad . \tag{1.6}$$

For further results of this type, refer to [15], [22] and [23].

In 1992 Sárközy [16] commenced the study of the multiplicative analogues of the above results, where in place of terms $a+b$ one considers terms $ab+1$. In particular, he proved the multiplicative analogue of (1.4). Let ϵ and δ be positive real numbers and let N be a positive integer. Let A be a subset of $\{1, \ldots, N\}$ of cardinality at least δN. He proved that there exists a positive number N_1, which is effectively computable terms of ϵ and δ, such that if N exceeds N_1 there exist integers a and a^1 from A such that

$$w(aa^1+1) > (1-\epsilon)(\log N)/\log\log\ N \quad . \tag{1.7}$$

We remark that this is slightly weaker than (1.4) since only the special case $A = B$ is covered and since while one cannot replace the factor $1-\epsilon$ in (1.4) by $1+\epsilon$ one expects (1.7) to hold with $2-\epsilon$ in place of $1-\epsilon$.

Our goal in this paper is to study the multiplicative analogues of (1.1), (1.2), (1.3), (1.5) and (1.6).

2 Lower bounds

Denote the set of positive integers by $\mathbb{N}$. We will prove the following multiplicative analogue of (1.2).

Theorem 1. *Let A and B be finite subsets of $\mathbb{N}$ with $|A| \geq |B| \geq 2$. Then*

$$w\left(\prod_{a\epsilon A, b\epsilon B}(ab+1)\right) > c_5 \log|A| \quad ,$$

where c_5 is an effectively computable absolute constant.

Both (1.2) and Theorem 1 are special cases of Theorem 2 below.

Theorem 2. *Let $n \geq 2$ be an integer, and let A and B be finite subsets of $\mathbb{N}^n$ with*
$|A| \geq |B| \geq 2(n-1)$ and with the following properties: the n-th coordinate of each vector in A is equal to 1 and any n vectors in $B \cup (0,\ldots,0,1)$ are linearly independent. Then

$$w\left(\prod_{\substack{(a_1,\ldots,a_n)\epsilon A\\(b_1,\ldots,b_n)\epsilon B}}(a_n b_1 + \cdots + a_n b_n)\right) > c_6 \log|A| \tag{2.1}$$

with an effectively computable positive number c_6.

(1.2) follows from Theorem 2 by taking $n = 2$ and $b_1 = 1$ for all $(b_1, b_2)\epsilon B$. Further, for $n = 2$ Theorem 2 gives Theorem 1 if $b_2 = 1$ for each $(b_1, b_2)\epsilon B$.

The next theorem is a slightly modified version of Theorem 2. A vector $\underline{a} = (a_1,\ldots,a_n)\epsilon\mathbb{N}^n$ is called primitive if $a_1,\ldots,a_n$ are relatively prime.

Theorem 3. *Let $m \geq 2$ be an integer, and let A and B be finite subsets of $\mathbb{N}^n$ with $A|A \geq |B| \geq 2n-1$ and with the following properties: A consists of primitive vectors and any n vectors in B are linearly independent. Then the lower estimate* (2.1) *holds.*

In Theorems 2 and 3 all assumptions are necessary. For example, the vectors $\underline{a}$ in A must be primitive, since otherwise the left-hand side of (2.1)

may assume the value

$$w\left(\prod_{(b_1,\ldots,b_n)\epsilon B}(a_1b_1+\cdots+a_nb_n)\right)$$

for each $\underline{a}=(a_1,\ldots,a_n)\epsilon A$. This is the case if A consists of vectors of the form
$p^m\underline{a}, m=1,2,\ldots,$ where p is a prime and $\underline{a}\epsilon\mathbb{N}^n$. Further, it is easy to see that the lower bounds $2(n-1)$ and $2n-1$, respectively, for $|B|$ cannot be lowered and that the linear independence of the vectors in B (resp. in $B\cup(0,\ldots,0,1)$) is necessary.

Since the n-th prime can be estimated from below by a constant times $n\log n$, Theorem 1 implies the following result.

Corollary 1. *Let A and B be finite subsets of $\mathbb{N}$ with $|A|\geq|B|\geq 2$. Then there exists a in A and b in B such that*

$$P(ab+1)>c_4\log|A|\log\log\ |A| \quad ,$$

where c_7 is an effectively computable positive constant.

Theorems 2 and 3 have similar consequences. An easy consequence of Theorem 1 is as follows.

Corollary 2. *Let A be a finite subset of $\mathbb{N}$ with $|A|\geq 2$. Then*

$$w\left(\prod_{\substack{a,a^1\epsilon A\\ a\neq a^1}}(a\cdot a^1+1)\right)>c_8\log|A| \quad ,$$

where c_8 is an effectively computable positive constant.

We remark that a similar lower bound can be given for the total number of distinct prime factors of the special numbers of the form a, a^1+1 with $a^1=a_1a\epsilon A$. For if $p_1,\ldots,p_s$ denotes the distinct prime factors of $\prod_{a\epsilon A}(a^2+1)$, then all $x=a\epsilon A$ satisfy the equation $x^2+1=p_1^{z_1}\cdots p_s^{z_s}$ in $x\epsilon\mathbb{N}, \mathrm{z}_1,\ldots,\mathrm{z}_\mathrm{s}\geq 0$. Now Theorem 2 of Evertse [6] gives $|A|\leq 3\cdot 7^{6+4s}$, whence

$$w\left(\prod_{a\epsilon A}(a^2+1)\right)>c_9\log|A|$$

follows with an effectively computable positive constant c_9. We note that this result has no additive analogue.

By Corollary 2 there exist distinct a, a^1 in A with $P(a, a^1 + 1) \to \infty$ as $|A| \to \infty$. This suggests the following conjecture.

Conjecture. *Let a, b and c denote distinct positive integers. If* $\max(a, b, c) \to \infty$ *then*

$$P((ab + 1)(bc + 1)(ca + 1)) \to \infty \quad .$$

To prove Theorems 2 and 3, we shall need two lemmas. Let

$$F(\boldsymbol{x}\ ) = F(x_1, \ldots, x_n) \epsilon \mathbb{Z}[x_1, \ldots, x_n]$$

be a decomposable form of degree r, that is a homogeneous polynomial which factorizes into linear forms $l_1(\boldsymbol{x}\ ), \ldots, l_r(\boldsymbol{x}\ )$ over a finite extension of $\mathbb{Q}$. Let R be a subring of $\mathbb{Q}$ which is finitely generated over $\mathbb{Z}$, and consider the decomposable form equation

$$F(\boldsymbol{x}\ ) \epsilon R^* \ \text{ in } \ \boldsymbol{x}\ = (x_1, \ldots, x_n) \epsilon R^n \tag{2.2}$$

where R^*denotes the multiplicative group of units of R. If $\boldsymbol{x}$ is a solution of (2.2) then so is $\epsilon\ \boldsymbol{x}$ for every $\epsilon \epsilon\ R^*$. A set of solutions of the form $R^*\ \boldsymbol{x}$ is called an R^*-coset of solutions.

In [8], Evertse and György gave a finiteness criterion for equation (2.2). In the special case when the splitting field of F is $\mathbb{Q}$ this criterion can be formulated in the following form. Denote by L_0 a maximal subset of pairwise linearly independent linear forms in $\{l_1, \ldots, l_r\}$. For any system L of linear forms from $\mathbb{Q}[x_1, \ldots, Z_n]$, we denote by $\nu(L)$ the $\mathbb{Q}$-vector space generated by the forms of L. Then we have

Lemma 1. *Suppose that the linear factors $l_1, \ldots, l_r$ of F have rational coefficients. Then the following two statements are equivalent:*

(i) *The forms in L_0 have rank n over $\mathbb{Q}$ and for each proper non-empty subset L_1 of L_0 there is a linear form in L_0 which is contained both in $\nu(L_1)$ and in $\nu(L_0 \backslash L_1)$;*

(ii) *The number of R^*-cosets of solutions of* (2.2) *is finite for every finitely generated subring R of $\mathbb{Q}$.*

Proof. This is an immediate consequence of Theorem 2 and the Proposition in [8].

Using a result of Schlickewei [19] on S-unit equations, Györy [10] gave an upper bound for the number of families of solutions of (2.2). This implies an upper bound for the number of R^*-cosets of solutions of (2.2), provided that condition (i) in Lemma 1 is fulfilled. Recently Evertse [7] has improved this latter bound by proving the following result.

Lemma 2. *If the finiteness condition* (i) *of Lemma 1 holds, then equation* (2.2) *has at most*

$$(2^{34}r^2)^{n^3(s+1)}$$

R^**-cosets of solutions.*

The proof depends on Evertse's improvement of the quantitative subspace theorems of Schmidt [21] amd Schlickewei [20].

Proof of Theorem 2. It suffices to prove the theorem for the case when B has cardinality $2(n-1)$. Put $r = 2n-1$. Let $\boldsymbol{b}_i = (b_{i1}, \ldots, b_{in})$ be the elements of B for $i = 1, \ldots, r-1$, and put $\boldsymbol{b}_r = (b_{r1}, \ldots, b_{rn}) - (0, \ldots, 0, 1)$. Let $l_i(\boldsymbol{x}) = b_{i1}x_1 + \cdots + b_{in}x_n$ for $i = 1, \ldots, r$. Then $F(\boldsymbol{x}) = l_1(\boldsymbol{x}) \cdots l_r(\boldsymbol{x})$ is a decomposable form of degree r with coefficients in $\mathbb{Z}$ which factorizes into linear factors over $\mathbb{Q}$. Denote by $p_1, \ldots, p_s$ the distinct prime factors of the product

$$\prod_{\substack{(a_1,\ldots,a_n)\in A \\ i=1,\ldots,r}} (a_1b_{i1} + \cdots + a_nb_{in}) \quad ,$$

and by R the ring $\mathbb{Z}\left[\frac{1}{p_1 \cdots p_s}\right]$. Then we have $s > 0$. Since, by assumption, $a_n = 1$ for all $(a_1, \ldots, a_n) \in A$, all the vectors $\underline{a} = (a_1, \ldots, a_n)$ in A are solutions of the decomposable form equation

$$F(\boldsymbol{x}) \in R^* \text{ in } \boldsymbol{x} = (x_1, \ldots, x_n) \in R^n \quad , \tag{2.3}$$

and these solutions belong to distinct R^*-cosets.

We use now an idea from the proof of Theorem 3 of [12]. Put $L_0 = \{l_1, \ldots, l_r\}$. By assumption, the forms in L_0 have rank n and are pairwise linearly independent over $\mathbb{Q}$. Consider an arbitrary proper non-empty subset L_1 of L_0. Since $r = 2n-1$, at least one of L_1 and $L_0 \backslash L_1$ has cardinality at least n. If $|L_1| \geq n$ then L_1 has rank n. In this case we have $L_0 \backslash L_1 \subseteq \nu(L_1)$ and so $L_0 \backslash L_1$ is contained both in $\nu(L_1)$ and in $\nu(L_0 \backslash L_1)$. If $|L_0 \backslash L_1| \geq n$, we get in the same way that L_1 is contained in $\nu(L_1)$ and $\nu(L_0 \backslash L_1)$. We

can now apply Lemma 1 and 2 to equation (2.3). We get

$$|A| \leq (2^{34}(2n-1)^2)^{n^3(s+1)} \quad .$$

Our result now follows by taking logarithms.

Proof of Theorem 3. Theorem 3 can be proved in a similar way as Theorem 2 above.

3 An upper bound

In this section we will prove the multiplicative analogue of (1.3). Erdös, Stewart and Tijdeman [4] proved a result which includes (1.3) as a special case. Let $\epsilon > 0$. For instance, it follows from Theorem 1 of [4] that there is a positive number c_{10} which is effectively computable in terms of ϵ, such that if k is an integer larger than c_{10} and l is an integer with $2 \leq l \leq (\log k)/\log\log\ k$ then there exists a set of positive integers A of cardinality k and a set of non-negative integers of cardinality l such that

$$P\left(\prod_{a\epsilon A}\prod_{b\epsilon B}(a+b)\right) < \left((1+\epsilon)\ \frac{\log k}{l}\log\left(\frac{\log k}{e}\right)\right)^l \quad . \tag{3.1}$$

In this section we shall prove the following result.

Theorem 4. *Let ϵ be a positive real number and let k and l be positive integers with*

$k \geq 3$ and $2 \leq l \leq \left(\dfrac{\log\log\ k}{\log\log\ \log k}\right)^{1/2}$. There exists a positive number $c_{11}(\epsilon)$, which is effectively computable in terms of ϵ, such that if k exceeds $c_{11}(\epsilon)$ then there are sets of positive integers A and B with $|A| = k$ and $|B| = l$ for which

$$P\left(\prod_{a\epsilon A}\prod_{b\epsilon B}(ab+1)\right) < (\log k)^{l+1+\epsilon} \quad . \tag{3.2}$$

Of course estimate (3.2) also applies with w in place of P. While the estimate (3.2) is weaker than (3.1) it is worth noting that we have allowed B to include 0 in the additive case and not in the multiplicative case. In the latter case we may certainly add 0 to B and so increase the cardinality of B by 1 without affecting tht upper bound. On the other hand, (3.1) applies

over a wider range for l. Indeed, Erdös, Stewart and Tijdeman were able to obtain significant improvements on the trivial estimate $k+l$ for l in the range $2 \le l \le \theta \log k$ for any real number θ less than 1, see Theorem 2 of [4]. We are able to extend the range for l in the statement of Theorem 4 and bound the largest elements of A and B at the cost of some precision in our upper bound in (3.2).

Theorem 5. *Let k and l be positive integers with $k \ge 3$. There exist effectively computable positive numbers c_{12} and c_{13} such that if k exceeds c_{12} and*

$$2 \le l \le c_{13}(\log k)/\log\log\ k \quad ,$$

then there are subsets A and B of $\{1,\dots,k^3\}$ with $|A| = k$ and $|B| = l$ for which

$$P\left(\prod_{a\epsilon A}\prod_{b\epsilon B}(ab+1)\right) < (\log k)^{5l} \quad . \tag{3.3}$$

One reason that the upper bound (3.2) and (3.3) are not as sharp as (3.1) is that we must replace Lemma 1 of [4] of Lemma 4 below.

Lemma 3 *Let N, L, t and l be positive integers with*

$$4lL \le t \quad , \tag{3.4}$$

Let S be a set of N elements and let $A_1,\dots,A_t$ be subsets of S with at least N/L elements. Then there exist distinct integers $i_1,\dots,i_l$ such that

$$|A_{i_1} \cap \cdots \cap A_{i_e}| \ge N/(4L)^l \quad .$$

Proof. Let $a_1,\dots,a_N$ be the elements of A and put

$$M = \max_{1\le i_1<i_2<\cdots<i_l\le t} |A_j \cap \cdots \cap A_{i_e}|$$

and

$$Z = \sum_{1\le i_1<\cdots<i_l\le t} |\ A_{i_1} \cap \cdots \cap A_{i_e}| \quad .$$

We have

$$Z \le M\left(\begin{array}{c} t \\ l \end{array}\right) \le Mt^l/l! \quad . \tag{3.5}$$

Further, on putting $N_j = |\ \{i \mid 1 \leq i \leq t, a_j \epsilon A_i\}|$ for $j = 1, \ldots, n$, we see that

$$\begin{aligned} Z &= \sum_{1 \leq i_1 < \cdots < i_l \leq t} \ \sum_{\substack{\tau \leq j \leq N \\ a_j \epsilon A_{i_1}}} \cap \cdots \cap A_{i_l} \\ &= \sum_{j=1}^{N} \ \sum_{\substack{1 \leq i_1 < \cdots < i_l \leq t \\ a_j \epsilon A_{i_1} \cap \cdots \cap A_{i_l}}} 1 \qquad\qquad = \sum_{j=1}^{N} \begin{pmatrix} N_j \\ l \end{pmatrix} \quad . \end{aligned} \tag{3.6}$$

We shall now estimate $\sum_{j=1}^{n} \begin{pmatrix} N_j \\ l \end{pmatrix}$ from below. To this end we note that

$$\sum_{j=1}^{N} N_j = \sum_{j=1}^{N} \sum_{\substack{1 \leq i \leq t \\ a_j \epsilon A_i}} = \sum_{i=1}^{t} \sum_{\substack{1 \leq j \leq N \\ a_j \epsilon A_j}} 1 = \sum_{j=1}^{t} |A_i| \quad ,$$

hence that

$$\sum_{j=1}^{n} N_j \geq Nt/L \quad . \tag{3.7}$$

Put

$$J = \{j \mid 1 \leq j \leq N, N_j > \frac{t}{2L}\} \quad .$$

We have, by (3.7),

$$\sum_{j \epsilon J} N_j = \sum_{j=1}^{N} N_j - \sum_{\substack{1 \leq j \leq N \\ j \notin J}} N_j \geq \sum_{j=1}^{N} N_j - \frac{Nt}{2L} \geq \frac{Wt}{2L} \quad . \tag{3.8}$$

Further, by (3.4), for all j in J,

$$\begin{pmatrix} N_j \\ l \end{pmatrix} = \frac{N_j(N_j - 1) \cdots (N_j - l + 1)}{l!} \geq \frac{(N_j/2)^2}{l!} \quad . \tag{3.9}$$

Since, for any positive real numbers $x_1, \ldots, x_u$,

$$\sum_{i=1}^{u} x_i^l \geq \left(\sum_{i=1}^{u} x_i \right)^l / u^{l-1} \quad ,$$

we have, from (3.8) and (3.9),

$$\sum_{j \epsilon J} \begin{pmatrix} N_j \\ l \end{pmatrix} \geq \frac{1}{2^l l!} \left(\frac{Nt}{2L} \right)^l N^{-l+1} = \frac{N}{(4L)^l} \ \frac{t^l}{e!} \quad . \tag{3.10}$$

Our result now follows from (3.5), (3.6) and (3.10).

Lemma 4. *Let N, L and l be positive integers with $l \leq L \leq N$ and let X and Y be non-empty sets of positive integers such that*

$$4lL \leq |X| \quad , \tag{3.11}$$

and for each x in X there are at least N/L integers j with $1 \leq j \leq N$ for which jx is in Y. Then there is a subset A of $\{1, \ldots, N\}$ and a subset B of X with

$$|B| = l \text{ and } |A| \geq N/(4L)^l \quad , \tag{3.12}$$

for which

$$A \cdot B \subset Y \quad .$$

Proof. We apply Lemma 3 with $S = \{1, \ldots, N\}$, $t = |X|$, $x = \{x_1, \ldots, x_t\}$ and $A_i = \{j \mid 1 \leq j \leq N \text{ and } jx_i \epsilon Y\}$ for $i = 1, \ldots, t$. Note that $|A_i| \geq N/L$ for $i = 1, \ldots, t$. Then there exist distinct integers $i, \ldots, i_l$ such that$|A_{i_1} \cap \cdots \cap A_{i_l}| \geq N/(4L)^l$. Put $A = A_{i_1} \cap \cdots \cap A_{i_l}$ and $B = \{x_{i_l}, \ldots, x_{i_l}\}$. Our result now follows.

Lemma 5. *Let M be an integer, N a positive integer and $a_{M+1}, \ldots, a_{M+N}$ complex numbers. For each character X we put*

$$T(\mathcal{X}) = \sum_{n=M+1}^{M+N} a_n \, \mathcal{X}(n) \quad .$$

Then for any $Q \geq 1$, we have

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\mathcal{X} (\text{mod q})}^{*} |T(\mathcal{X})|^2 \leq (Q^2 + \pi N) \sum_{n=M+1}^{M+N} |a_n|^2 \quad ,$$

where $\sum_{\mathcal{X} (\text{mod q})}^{*}$ denotes a sum over all primitive characters modulo q.

Proof. This character version of the large sieve is due to Gallagher [9].

Lemma 6. *Let R be a positive integer, J a subset of $\{1, \ldots, R\}$ and Q a real number with $Q \geq 1$. For each prime p, denote the number of solutions of the congruence*

$$rr^1 \equiv 1 (\text{mod p}) \quad ,$$

with r and r^1 in 5, by $F(5,p)$ and denote the number of the integers in J divisible by p by $G(J,p)$. Then

$$\sum_{p\leq Q} p \mid F(J,p) - \frac{1}{p-1}\ (|J| - G(J,p)^2| \leq (Q^2 + \pi R)|J| \quad .$$

Proof. Let $\mathcal{X}_0$ denote the principal character modulo p. We have

$$\begin{aligned} F(J,p) &= \sum_{r\epsilon J} \sum_{r^1\epsilon J} \frac{1}{\varphi(p)} \sum_{\mathcal{X}(\text{mod p})} \mathcal{X}(rr^1) \\ &= \frac{1}{p-1} \sum_{\mathcal{X}(\text{mod p})} \left(\sum_{r\epsilon J} \mathcal{X}(r)\right)^2 \\ &= \frac{1}{p-1} \left(\left(\sum_{\substack{r\epsilon J \\ p\not| r}} 1\right)^2 + \sum_{\mathcal{X}\neq\mathcal{X}(\text{mod p})} \left(\sum_{r\epsilon J} \mathcal{X}(r)\right)^2 \right) \\ &= \frac{1}{p-1} \left((|J| - G(J,p))^2 + \sum_{\mathcal{X}(\text{mod p})}^{*} \left(\sum_{r\epsilon J} \mathcal{X}(r)\right)^2 \right) \end{aligned}$$

$$\left| F(J,p) - \frac{1}{p-1}(|J| - G(J_p))^2 \right| \leq \frac{1}{p-1} \sum_{\mathcal{X}(\text{mod p})}^{*} \left| \sum_{r\epsilon J} \mathcal{X}(r) \right|^2 \quad .$$

By Lemma 5, it follows that

$$\sum_{p\leq Q} p \left| F(J,p) - \frac{1}{p-1}\ (|J| - G(J,p))^2 \right| \leq (Q^2 + \pi R)|J| \quad .$$

Let $\psi(x,y)$ be the number of positive integers not exceeding x which are free of prime divisors larger than y.

Lemma 7. *Let x be a positive integer and u a real number with $a \geq 3$. There exists an effectively computable constant c_{14} such that*

$$\psi(x, x^{1/u}) \geq x \exp\left(-u\left(\log u + \log\log\ u - 1 + c_{14}\left(\frac{\log\log\ u}{\log u}\right)\right)\right) \quad .$$

Proof. See Theorem 3.1 of Canfield, Erdös and Pomerance [1].

For any positive integer n let $\tau(n)$ denote the number of positive divisors of n.

Lemma 8. *There is an effectively computable number c_{15} such that if N is a positive integer larger than c_{15} and A is a subset of $\{1,\dots,N\}$ then the set $A^1 = \{a \mid a\epsilon A \text{ and } \tau(a) < (4N\log N)/|A|\}$ satisfies*

$$|A^1| > |A|/2 \quad . \tag{3.13}$$

Proof. There is an effectively computable number N_0 such that for $N > N_0$,

$$\sum_{a\epsilon A} \tau(a) \leq \sum_{n=1}^{N} \tau(n) < 2N\log N \quad , \tag{3.14}$$

see, for instance, Theorem 320 of [13].

On the other hand, we have

$$\sum_{a\epsilon A} \tau(a) \geq \sum_{a\epsilon (A\setminus A)^1} \tau(a) \geq \sum_{a\epsilon (A\setminus A^1)} (4N\log N)/|A|$$

so

$$\sum_{a\epsilon A} \tau(a) \geq 2N\log N(2 - 2|A^1|/|A|) \quad . \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$2 - 2|A^1|/|A| < 1$$

and this implies (3.13).

Proof of Theorem 4. We may assume, without loss of generality, that $0 < \epsilon < 1$. Let $c_1, c_2, \dots$ denote positive numbers which are effectively computable in terms of ϵ. Let N be a positive integer larger than 30 and let l be a positive integer with

$$2 \leq l \leq ((\log\log\ N)/\log\log\ \log N)^{1/2} \quad . \tag{3.16}$$

For any real number x let $[x]$ denote the greatest integer less than or equal to x. Put $R = [N^{(e+1)/2e}], Q = 2N^{1/e}$ and $y = (\log R)^{l+1+\epsilon}$. Let J denote the set of positive integers n with $n \leq R$ and $P(n) \leq y$. Put

$$u = \frac{\log R}{(lH+\epsilon)\log\log\ R} \quad ,$$

and notice that for $N > c_1$, $u \geq 3$, hence, by Lemma 7,

$$|J| \geq \psi(R,Y) \geq R\exp\left(-u\left(\log u + \log\log\ u - 1 + c_{15}\left(\frac{\log\log\ u}{\log u}\right)\right)\right) \quad . \tag{3.17}$$

Thus, for $N > c_2$,

$$|J| \geq R^{1-\frac{1}{l+1+\epsilon}} = R^{\frac{e}{e+1}+\frac{\epsilon}{2(l+1)^2}} \quad ,$$

whence

$$|J| \geq N^{1/2}\ N^{\epsilon/(3l(l+1))} \quad , \tag{3.18}$$

for $N > c_3$.

Let F be the set of integers of the form $rr^1 - 1$ with r, r^1 in J. Define $F(J,p)$ to be the number of pairs (r, r^1) with $rr^1 - 1$ divisible by p and let $G(J,p)$ be the number of integers in J divisible by p.

Let E be the set of prime p with $Q/2 < p \leq Q$ for which

$$F(J,p) > |J|^2/2Q \quad , \tag{3.19}$$

and let E be the other primes in this range. Observe that for $N > c_4$,

$$y < Q/2 \quad ,$$

so $G(J,p) = 0$ whenever p exceeds $Q/2$. Thus for $p \epsilon \overline{E}$ we have

$$\frac{1}{p-1}\ (|J| - G(J,p))^2 = \frac{|J|^2}{p-1} \geq \frac{|J|^2}{Q} \quad . \tag{3.20}$$

From Lemma 6, we deduce that

$$\sum_{p \epsilon \overline{E}} p \mid F(J,p) - \frac{1}{p-1}(|J| - G(J,p))^2 \mid \ \leq (Q^2 + \tau R)|J| \quad .$$

Since for $p \epsilon \overline{E}$ we have, by (3.18) and (3.19),

$$|F(J,p) - \frac{1}{p-1}\ (|J| - G(J,p))^2 \mid \ > |J|^2/2Q \quad ,$$

it follows that

$$|\overline{E}|\ |J|^2/4 \leq (Q^2 + \pi R)|J| \quad , \tag{3.21}$$

hence that

$$|\overline{E}| \leq 32 \max\left(\frac{N^{2/l}}{|J|}\ ,\ \frac{R}{|J|}\right) \quad .$$

Thus, by (3.18),

$$|\overline{E}| \leq \begin{cases} N^{1/2}N^{-\epsilon/20} & \text{for } l = 2 \\ N^{1/2l} & \text{for } 1 \neq 2 \end{cases} \quad ,$$

for $N > c_5$. However, for $N > c_6$, there are at least $Q/(3 \log Q)$ prime p with $Q/2 < p \leq Q$. Further, for $N > c_7$,

$$|\overline{E}| < Q/(6 \log Q) \quad ,$$

whence

$$|E| > Q/(6 \log Q) \quad . \tag{3.22}$$

For each prime p in E there are more than $|J|^2/2Q$ pairs (r, r^1) with r and r^1 in R for which p divides $rr^1 - 1$. Put $D = \max_{n \leq R} \tau(r)$. By, for instance, Theorem 317 of [13],

$$D < \exp(\log N / \log \log \ N) \quad ,$$

for $N > c_8$. Moreover, if an integer n can be represented in the form rr^1 with r and r^1 in R then it can be represented in at most D^2 ways in this form. Thus, for each prime p in E there are at least $|J|^2/2D^2Q$ distinct integers f with $f = rr^1 - 1$ and for which p divides f. Let $j = f/p$ and notice that

$$1 \leq j \leq R^2/(Q/2) < N \quad .$$

For $N > c_9$, we have

$$|J|^2/2D^2Q \geq N/L \quad ,$$

where

$$L = \frac{1}{4} \ N^{(1/l) - \epsilon/(4l(l+1))} \quad . \tag{3.23}$$

We may now apply Lemma 4 with $X = E$ and $Y = F$. We remark that condition (3.11) applies for $N > c_{10}$ by virtue of (3.22) and (3.23). We find that there is a subset A_1 of $\{1, \ldots, N\}$ and a subset B of E with $|B| = l$ and

$$|A_1| \geq N/(4l)^l = N^{\epsilon/4(l+1)} \quad ,$$

for which $A_1 \cdot B$ is contained in F.

Let k be an integer larger than 3 and let l be an integer with $2 \leq l \leq \left(\dfrac{\log \log \ k}{\log \log \ \log k} \right)^{1/2}$. Choose N so that

$$k = [N^{\epsilon/4(l+1)}] \quad .$$

Since $1 \leq N$, (3.16) holds and provided that k exceeds c_{11}, we may find A_1 and B as above. Let A be a subset of A_1 with $|A| = k$. Notice that

$$(\epsilon/5(l+1)) \log N < \log k$$

for $N > c_{12}$ and that

$$\log R \leq ((l+1)/2l) \log N \quad .$$

Thus, for $k > c_{13}$, we have

$$\begin{aligned} y &\leq ((5(l+1)^2/2\epsilon l) \log k)^{l+1+\epsilon} \\ &\leq (\log k)^{l+1+2\epsilon} \end{aligned} \quad .$$

Since $P(ab+1)$ is at most y whenever a is in A and b is in B, our result follows.

Proof of Theorem 5. Our proof of Theorem 5 is a modification of the proof of Theorem 4. Let $c_1, c_2, \ldots$ denote effectively computable positive numbers. Let k be a positive integer, let θ be a positive real number and let l be an integer with

$$2 \leq l \leq (\theta \log k)/\log\log\ k \quad . \tag{3.24}$$

Put $N = k_1^3\ Q = 2N^{1/2}$ and $R = [N^{3/4}]$. Let

$$y = (\log R)^{14l/3} \tag{3.25}$$

and put

$$u = (14 \log R)/3l \log\log\ R \quad .$$

Let J^1 denote the set of positive integers n with $n \leq R$ and $P(n) \leq y$. If $\theta < c_1$ we have $u \geq 3$ and so (3.17) holds with J^1 in place of J. Further if $\theta < c_2$ we have

$$-1 + c_{15}((\log\log\ u)/\log u) < 0 \quad ,$$

and so, for $k > c_3$,

$$|J^1| \geq 2N^{3/4(1-3/14l)} \quad .$$

We may now apply Lemma 7 to find a subset J of J^1 with $|J| \geq |J^1|/2$, hence for which

$$|J| \geq N^{3/4(1-3/14l)} \quad , \tag{3.26}$$

and for which D, the maximum of $\tau(r)$ for n in J, satisfies

$$D < 4R \log R/|J^1| \quad .$$

Thus, for $k > c_4$;

$$D < 2N^{9/56l} \log N \quad . \tag{3.27}$$

We now define F, E and $\overline{E}$, as in the proof of Theorem 4. We again apply Lemma 6 to deduce that (3.21) holds. Consequently, for $k > c_5$, we find that

$$|\overline{E}| \leq 20N/|J| \quad ,$$

and, from (3.26), we see that

$$|\overline{E}| < Q/6 \log Q \quad ,$$

whence (3.22) holds.

Therefore, as in the proof of Theorem 4, we find that there are at least $|J|^2/2D^2Q$ distinct ingegers f with $f = rr^1 - 1, r$ and r^1 in J, and for which p divides f. Let $j = f/p$ and notice that $1 \leq j \leq N$. Further, we have

$$|J|^2/2D^2Q \geq N/16N^{36/56l}(\log N)^2 \quad .$$

Thus, for $\theta < c_6$ and $k > c_7$, we have

$$|J|^2/2D^2Q \geq N/L \quad ,$$

where

$$L = \frac{1}{4}\, N^{2/3l} \quad . \tag{3.28}$$

We may now apply Lemma 4 with $X = E$ and $Y = F$. For $\theta < c_8$, (3.11) holds by virtue of (3.24) and (3.28). We find that there is a subset A_1 of $\{1, \ldots, N\}$ and a subset B of E with $|B| = l$ and

$$|A_1| \geq N/(4l)^l = N^{1/3} \quad ,$$

for which $A_1 \cdot B$ is contained in F. We now let A be a subset of A_1 with $|A| = k$. Take $\theta = \frac{1}{2}\min(c_1, c_2, c_6, c_8)$. Then for $k > c_9$, (3.24) holds and

$$P\left(\prod_{a \epsilon P}\ \prod_{b \epsilon B}(ab+1)\right) < \left(\frac{9}{4}\log k\right)^{14l/3} < (\log k)^{5l} \quad ,$$

as required.

4 Terms with many prime factors

In this section we shall establish the multiplicative analogue of (1.5). For the proof we shall require the following result which is a consequence of the large sieve inequality.

Lemma 9. *Let N be a positive integer and let A and B be non-empty subsets of $\{1,\dots,N\}$. Let α and β be real numbers with $\alpha > 1$. Let T be the set of primes p which satisfy*
$\beta < p \le (\log N)^{\alpha}$ and let S be a subset of T consisting of all but at most $2\log N$ elements of T. There is a real number c_{16}, which is effectively computable in terms of α and β, such that if N exceeds c_{16} and

$$(|A||B|)^{1/2} \ge N^{(1+1/2)/2}/10$$

then there is a prime p from S and elements a from A and b from B such that p divides $ab+1$.

Proof. This is Lemma of [18].

We shall use Lemma 9 to prove the next result.

Theorem 6. *Let θ be a real number with $1/2 < \theta \le 1$ and let N be a positive integer. There exists a positive number c_{17}, which is effectively computable in terms of θ, such that if A and B are subsets of $\{1,\dots,N\}$ with N greater than c_{17} and*

$$(|A||B|)^{1/2} \ge N^{\theta} \quad , \tag{4.1}$$

then there exists an integer a from A and an integer b from B for which

$$w(ab+1) > \frac{1}{6}\ (\theta = 1/2)^2 \log N/\log\log\ N \quad . \tag{4.2}$$

Proof. Our proof is very similar to the proof of Theorem 1 of [17]. We have repeated parts of that argument here for the convenience of the reader.

Let $\theta_1 = (\theta + 1/2)/2$ and define G and v by

$$G = (\log N)^{1/(2\theta_1 - 1)} \quad ,$$

and

$$v = \left[\frac{1}{6}\ (\theta - 1/2)^2 \frac{\log N}{\log\log\ N}\right] + 1 \quad , \tag{4.3}$$

respectively.

Put $A_0 = A, B_0 = B$ and $W_0 = \phi$. We shall construct inductively sets $A_1,\dots,A_v$, $B_1,\dots,B_v$ and $W_1,\dots,W_v$ with the following properties. First, W_i is a set of i primes q satisfying $10 < q \le G$, $A_i \subseteq A_{i-1}$ and $B_i \subseteq B_{i-1}$ for $i = 1,\dots,v$. Secondly, every element of the set $A_iB_i + 1$ is divisible by each prime in W_i for $i = 1,\dots,v$. Finally,

$$|A_i| \ge \frac{|A|}{G^{3i}} \ \text{ and } \ |B_i| \ge \frac{|B|}{G^{3i}} \quad , \tag{4.4}$$

for $i = 1, \ldots, v$. Note that this suffices to prove our result since A_v and B_v are both non-empty and on taking a from A_v and b from B_v we find that $ab + 1$ is divisible by the v primes from W_v and so (4.2) follows from (4.3).

Suppose that i is an integer with $0 \leq i < v$ and that A_i, B_i and W_i have been constructed with the above properties. We shall now show how to construct A_{i+1}, B_{i+1} and W_{i+1}. First, for each prime p with $10 < p \leq G$ let $a_1, \ldots, a_{j(p)}$ be representatives for those residue classes modulo p which are occupied by fewer than $|A_i|/p^3$ terms of A. For each prime p with $10 < p \leq G$ we remove from A_i those terms of A_i which are congruent to one of $a_1, \ldots, a_{j(p)}$ modulo p. We are left with a subset A_i^1 of A_i with

$$|A_i^1| \geq |A_i| \left(1 - \sum_{10<p\leq G} \frac{j(p)}{p^3}\right) \geq |A_i| \left(1 - \sum_{10<p} \frac{1}{p^3}\right) \geq \frac{|A_i|}{10} \tag{4.5}$$

and such that for each prime p with $10 < p \leq G$ and each a^1 in A_i^1, the number of terms of A_i which are congruent to a^1 modulo p is at least $|A_i|/p^3$. Similarly, we produce a subset B_i^1 of B_i with

$$|B_i^1| \geq |B_i|/10 \tag{4.6}$$

and such that for each prime p with $10 < p \leq G$ and each residue class modulo p which contains an element of B_i^1 the number of terms of B_i in the residue class is at least $|B_i|/p^3$.

The number of terms in W_i is i which is less than v and, by (4.3), is at most $\log N$. Further by (4.4), we find that

$$(|A_i|\ |B_i|)^{1/2} = (|A|\ |B|)^{1/2} G^{-3i} \geq N^{\theta_1} \tag{4.7}$$

Therefore, by (4.5), (4.6) and (4.7),

$$(|A_i^1|\ |B_i^1|)^{1/2} \geq N^{\theta_1}/10 \quad .$$

We now apply Lemma 9 with $A = A_i^1$, $B = B_i^1$, $B = 10$, $\alpha = 1/(\theta - 1/2)$ and S the set of primes p with $10 < p \leq G$ for which p is not in W_i. We find that provided that N exceeds a number which is effectively computable in terms of θ, there is a prime q_{i+1} in S, an element a^1 in A_i^1 and an element b^1 in B_i^1 such that q_{i+1} divides $a^1 b^1 + 1$. We put

$$A_{i+1} = \{a \epsilon A_i \quad : \quad a \equiv a^1 (\text{mod } \mathrm{q_{i+1}})\} \quad ,$$

$$B_{i+1} = \{b \epsilon B_i \quad : \quad b \equiv b^1 (\text{mod } \mathrm{q_{i+1}})\} \quad ,$$

and

$$W_{i+1} = W_i \cup \{q_{i+1}\} \quad .$$

By our construction every element of $A_{i+1}B_{i+1}+1$ is divisible by each prime in W_{i+1}. Further, we have, by (4.4),

$$|A_{i+1}| \geq \frac{|A_i|}{q_{i+1}^3} \geq \frac{|A_i|}{6^3} \geq \frac{|A|}{6^{3(i+1)}}$$

and

$$|B_{i+1}| \geq \frac{|B|}{6^{3(i+1)}} \quad ,$$

as required. Our result now follows.

5 Terms with few prime factors

Let N and l be positive integers with $l < \log N$. Pomerance, Sárközy and Stewart [14] proved that there exists an effectively computable positive number c_{18} such that if N exceeds c_{18} then there exist subsets A and B of $\{1, \ldots, N\}$ with $|B| = 1$ and

$$|A| > \frac{N}{l(\log N)^l} \quad ,$$

such that every element of $A + B$ is prime. We shall prove the following result.

Theorem 7 Let N and l be positive integers with

$$l \leq \frac{\log N}{2 \log\log N} \quad . \tag{5.1}$$

For N sufficiently large, there exists a set B of l prime numbers from $\{1, \ldots, [(\log N)^3]\}$ and a subset A of $\{1, \ldots, N\}$ with

$$|A| \geq \frac{N}{(8 \log N)^l} \quad ,$$

such that $ab + 1$ is a prime whenever a is from A and b is from B.

The proof depends on the Siegel-Walfiz theorem for primes in arithmetical progressions and as a consequence is ineffective in nature. In particular, we are not able to replace the requirement that N be sufficiently large with

the requirement that N be larger than an effectively computable positive number.

Let ϵ be a positive real number. It follows from Theorem 6 that if A and B are subsets of $\{1,\dots,N\}$ with $|A||B| > N^{1+\epsilon}$ then

$$\max_{a\epsilon A, b\epsilon B} w(ab+1) \to \infty \tag{5.2}$$

as $N \to \infty$. Taking $l = 2$ in the statement of Theorem 2 we see that there are subsets A and B of $\{1,\dots,N\}$ with $|B| = 2$ and

$$|A| \geq \frac{N}{64(\log N)^2} \quad ,$$

for which

$$\max_{a\epsilon A, b\epsilon B} w(ab+1) = 1 \quad . \tag{5.3}$$

Thus if we measure the size of A and B in terms of the geometric mean of the cardinalities of A and B we have determined, up to a factor of ϵ, when (5.2) holds. On the other hand if we measure the size of A and B in terms of the minimum of $|A|$ and $|B|$ a different situation applies. Certainly (5.2) holds if

$$\min(|A|,|B|) > N^{1/2+\epsilon} \tag{5.4}$$

by Theorem 6. Further, by Theorem 7 we see that there are subsets A and B of $\{1,\dots,N\}$ with

$$\min(|A|,|B|) \geq \left[\frac{\log N}{2\log\log\ N}\right] \quad , \tag{5.5}$$

for which (5.3) holds. There is a large gap between (5.4) and (5.5). We suspect that (5.5) is closer to the truth.

Proof of Theorem 7. Let X denote the set of prime numbers less than $(\log N)^3$. By the prime number theorem we have

$$|X| > \frac{(\log N)^3}{4\log\log\ N} \quad ,$$

for N sufficiently large. Let Y denote the set of integers of the form $p-1$ where p is a prime. By the Seigel-Walfisz theorem (see for example [2], p. 133) if q is in X then the number of integers j with $1 \leq j \leq N$ for which qj is in Y, or equivalently for which $qj+1$ is prime, is $(1+0(1))\dfrac{qN}{(q-1)\log N}$ and so for N sufficiently large exceeds N/L where $L = 2[\log N]$. We may now apply Lemma 4 with l satisfying (5.1). Then (3.11) holds for N sufficiently large and our result follows directly.

6 The average value of $w(ab+1)$

Finally, we shall prove the multiplicative analogue of (1.6).

Theorem 8. *There exists an effectively computable positive number c_{19} such that if T and N are positive integers with $T \le N^{1/2}$ and A and B are non-empty subsets of $\{1,\dots,N\}$ then*

$$\left| \frac{1}{|A|\ |B|} \sum_{T<p} \sum_{a\epsilon A, b\epsilon B, p|ab+1} 1 - (\log\log\ N - \log\log\ 3T) \right| < q_9 \left(1 + \frac{N}{T\min(|A|,|B|)}\right) \quad .$$

Taking $T = [N/\min(|A|,|B|)]$ in Theorem 8 we obtain the following result.

Corollary 3. *There exists an effectively computable positive number c_{20} such that if N is a positive integer and A and B are non-empty subsets of $\{1,\dots,N\}$ then*

$$\left| \frac{1}{|A||B|} \sum_{T,p} \sum_{a\epsilon A, b\epsilon B, p|ab+1} 1 - (\log\log\ N - \log\log\ 3T) \right| < q_9 \left(\frac{N}{T\min(|A|,|B|)}\right) \quad .$$

Taking $T = [N/\min(|A|,|B|)]$ in Theorem 8 we obtain the following result.

Corollary 3. *There exists an effectively computable positive number c_{20} such that if N is a positive integer and A and B are non-empty subsets of $\{1,\dots,N\}$ then*

$$\left| \frac{1}{|A||B|} \sum_{p>N/\min(|A|,|B|)} \sum_{a\epsilon A, b\epsilon B} 1_{p|ab+1} - (\log\log\ N - \log\log\ (3N/\min(|A|,|B|)) \right| < c_{20} \quad .$$

Therefore

$$\frac{1}{|A|\ |B|} \sum_{a\epsilon A} \sum_{b\epsilon B} w(ab+1) > (1+0(1))\log\log\ N \quad ,$$

provided that A and B are subsets of $\{1,\dots,N\}$ with

$$\min(|A|,|B|) = N\exp(-(\log N)^{0(1)}) \quad .$$

Proof of Theorem 8. The proof will be similar to the proof of Theorem 3 of [17]. However, while in [17] the crucial tool in the proof is the standard analytical form of the large sieve, here, due to the multiplicative structure of the numbers studies, we employ Lemma 5. Let $c_1, c_2, \ldots$ denote effectively computable positive numbers.

Put $R = [(n^2+1)^{1/4}]$. We have

$$\left| \sum_{a \epsilon A} \sum_{b \epsilon B} \sum_{T<p, p|ab+1} 1 - \sum_{a \epsilon A} \sum_{b \epsilon B} \sum_{T<p \leq R, p|ab+1} 1 \right| = \left| \sum_{a \epsilon A} \sum_{b \epsilon B} \sum_{R<p \leq N^2+1, p|ab+1} 1 \right|$$

$$\leq \left| \sum_{a \epsilon A} \sum_{b \epsilon B} 3 \right| = 3|A| \ |B| \quad . \tag{6.1}$$

We define, for each character χ,

$$F(\chi) = \sum_{a \epsilon A} \chi(a) \quad , \quad G(\chi) = \sum_{b \epsilon B} \chi(b) \quad .$$

Then

$$\sum_{a \epsilon A} \sum_{b \epsilon B} \sum_{T<p \leq R, p|ab+1} 1 = \sum_{T<p \leq R} \frac{1}{p-1} \overline{\chi}(-1) \sum_{a \epsilon A} \sum_{b \epsilon B} \chi(a,b)$$

$$= \sum_{T<p \leq R} \frac{1}{p-1} \left(\sum_{p \not| a, a \epsilon A} \sum_{p \not| b, b \epsilon B} + \sum_{\chi \neq \chi_0 (\text{mod p})} \overline{\chi}(-1) F(\chi) G(\chi) \right)$$

whence

$$\left| \sum_{a \epsilon A} \sum_{b \epsilon B} \sum_{T<p \leq R, p|ab+1} - |A| \ |B| \sum_{T<p \leq R} \frac{1}{p-1} \right|$$

$$\leq \sum_{T<p \leq R} \frac{1}{p-1} \left(\sum_{p|a, a \epsilon A} \sum_{b \epsilon B} 1 + \sum_{a \epsilon A} \sum_{p|b, b \epsilon B} 1 + \sum_{\chi \neq \chi_0 (\text{mod p})} |F(\chi)| G(\chi) \right)$$

$$\leq \sum_{T<p \leq R} \frac{1}{p-1} \left(\left(\sum_{p|n, n \leq N} 1 \right) (|A| + |B|) + \frac{1}{2} \sum_{\chi \neq \chi_0 (\text{mod p})} \left(|F(\chi)^2| + |G(\chi)|^2 \right) \right)$$

$$\leq 2 \left(|A| + |B| \right) \sum_{T<p \leq R} \frac{N}{p^2} + \sum_{T<p \leq R} \frac{1}{\varphi(p)} \sum_{\chi \neq \chi_0 (\text{mod p})} \left(|F(\chi)|^2 + |G(\chi)|^2 \right) \Bigg) .$$

Further, we have

$$\left| \sum_{T<p\leq R} \frac{1}{p-1} - (\log\log\ R - \log\log\ 3T) \right| < c_1 \quad .$$

Thus it follows that

$$\left| \sum_{a\epsilon A} \sum_{b\epsilon B} \sum_{T<p\leq R, p|ab+1} 1 - |A|\ |B|(\log\log\ R - \log\log\ 3T) \right|$$

$$< c_1|A|\ |B| + c_2 \frac{N}{T\log T}(|A|+|B|) + \sum_{T<p\leq R} \frac{1}{\varphi(p)} \sum_{\chi\neq\chi_0(\text{mod p})} (|F(\chi)|^2+|G(\chi)|^2) \quad . \tag{6.2}$$

Put

$$S(n) = \sum_{p\leq n} \frac{p}{\varphi(p)} \sum_{\chi\neq\chi_0(\text{mod p})} |F(\chi)|^2 \quad .$$

Then, by Lemma 5, for $n \leq R$ we have

$$S(n) \leq (n^2 + \tau N)|A| \leq 6N|A| \quad .$$

Thus we obtain by partial summation that

$$\begin{aligned} \sum_{T<p\leq R} \frac{1}{\varphi(p)} \sum_{\chi\neq\chi_0(\text{mod p})} |F(\chi)|^2 &= \sum_{n=T+1}^{R} \frac{S(n)-S(n-1)}{n} \\ &= \sum_{n=T+1}^{R} S(n)\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{S(T)}{T+1} + \frac{S(R)}{R+1} \\ &\leq \sum_{n=T+1}^{R} 6N(A)\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{6N|A|}{R+1} = \frac{6N|A|}{T+1} \quad , \end{aligned} \tag{6.3}$$

and similarly,

$$\sum_{T<p\leq R} \frac{1}{\varphi(p)} \sum_{\chi\neq\chi_0(\text{mod p})} |G(\chi)|^2 \leq \frac{6N|B|}{T+1} \quad . \tag{6.4}$$

It follows from (6.1), (6.2), (6.3) and (6.4) that

$$\left| \frac{1}{|A|\ |B|} \sum_{T<p} \sum_{a\epsilon A, b\epsilon B, |ab+1} 1 - (\log\log\ R - \log\log\ 3T) \right|$$

$$< c_3 \left(1 + \frac{N}{T}\left(\frac{1}{|A|} + \frac{1}{|B|}\right)\right) \quad ,$$

whence the result follows.

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