ALGEBRA COMPREHENSIVE EXAM, WINTER 2017

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Do all questions.

1. (a) If G is a non-abelian group of order p^3 (p prime), prove that the centre of G is the subgroup generated by all elements $aba^{-1}b^{-1}$ with $a, b \in G$.
(b) If G is a non-abelian group of order p^3 for an odd prime p, prove that G has exactly $p^2 + p - 1$ distinct conjugacy classes.

2. How many maximal ideals of $\mathbb{Z}[x]$ contain $\{30,\ x^2+1\}$?

3. If V is an inner product space over $\mathbb{R}$ or $\mathbb{C}$, a **rigid motion** is any function T from V to V (not necessarily linear) such that $\|T\alpha - T\beta\| = \|\alpha - \beta\|$ for all α, β in V. Recall that a linear operator T is called **unitary** if $\|T\alpha\| = \|\alpha\|$ for all α in V. A function S from V to V is called a **translation** if there exists $\gamma \in V$ such that $S\alpha = \alpha + \gamma$ for all α in V.
(a) Let T be a rigid motion such that $T(\mathbf{0}) = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in V. Show that T is linear and a unitary operator.
(b) Use the result of Part (a) to prove that every rigid motion is a translation followed by a unitary operator.
(c) Let $V = \mathbb{R}^2$ with the standard inner product over $\mathbb{R}$. Show that a rigid motion of $\mathbb{R}^2$ is either a translation followed by a rotation, or a translation followed by a reflection followed by a rotation.

4. (a) Give an example (with proof) of an irreducible polynomial in $\mathbb{Q}[x]$ of degree 6.
(b) Suppose $f(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ of degree $2n$. Prove that if E is a field extension of $\mathbb{Q}$ degree 2, then $f(x)$ is either irreducible in $E[x]$, or $f(x)$ factors in $E[x]$ as a product of two irreducible factors each of degree n.

5. Let F be a field. Show that

$$G = \left\{ \begin{bmatrix} x & a & b \\ 0 & y & c \\ 0 & 0 & z \end{bmatrix} \middle| \, x, y, z, a, b, c \in F; xyz \neq 0 \right\},$$

with the matrix product, is a solvable group.

6. Suppose R is a unital ring and M is a simple R-module. Prove that the additive group of M is either a direct sum of copies of $\mathbb{Q}$, or a direct sum of copies of $\mathbb{Z}_p$ for some prime p.

7. (a) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. Find the Jordan canonical form J of A and an invertible matrix P such that $A = P^{-1}JP$.
(b) Let M be an $n \times n$ complex matrix. Define the exponential e^M of M by

$$e^M = I_n + M + \frac{1}{2!}M^2 + \cdots + \frac{1}{l!}M^l + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!}M^i,$$

where I_n is the $n \times n$ identity matrix and $M^0 = I_n$. Compute e^A, where A is defined in (a).
(c) Prove that for any $n \times n$ complex matrix B, e^B exists (i.e., the infinite sum converges) and is invertible.

8. Prove or disprove the following: if F, K are fields with $\mathbb{Q} \leq F \leq K \leq \mathbb{C}$ and $[K : F] = 4$, then there exists an intermediate field strictly between F and K.

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