Department of Pure Mathematics February 4, 2015
Algebra Comprehensive Examination
Prepared by Rahim Moosa and Frank Zorzitto

The eight questions have equal weight. Attempt all of the questions.

1. Let $T : V \to V$ be a linear operator on an n-dimensional, complex vector space. Prove that there exists a tower of T-invariant subspaces

$$(0) \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n = V$$

such that $\dim V_j = j$ for every j. Deduce from this that V contains a basis such that the matrix of T using this basis is upper triangular. Do not invoke the Jordan canonical form theorem in your proof.

2. (a) Explain the class equation for a finite group, and illustrate it in the case of the group $\mathcal{S}_4$ of permutations on 4 letters.
   (b) Prove that every group of order p^2 is abelian.

3. (a) Show that $1 + \sqrt{-5}$ is irreducible but not prime in the ring $\mathbb{Z}\left[\sqrt{-5}\right]$.
   (b) Prove that the ideal $\left(2, 1 + \sqrt{-5}\right)$ is maximal in $\mathbb{Z}\left[\sqrt{-5}\right]$.

4. (a) Let K be a Galois extension of degree 12 over a field F. Prove that there is an intermediate field E between F and K such that E has degree 3 over F.
   (b) If ζ is a primitive 13th root of unity, how many subfields does the field $\mathbb{Q}(\zeta)$ contain? Justify your answer.
   (c) If $f(X)$ in $\mathbb{Q}[X]$ is an irreducible polynomial of degree $n \geq 2$, with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{C}$, show that $\displaystyle\sum_{j=1}^{n} \frac{1}{\alpha_j^2} \in \mathbb{Q}$.

5. (a) Let V be the vector space $P_3(\mathbb{R})$ of polynomials of degree ≤ 3 with coefficients in $\mathbb{R}$. Let $T : V \to V$ be the linear operator $T(p(x)) = xp''(x)$. Find the Jordan canonical form of T and a corresponding basis for V.

   (b) If two 3×3 matrices over the complex numbers have the same characteristic and the same minimal polynomial, prove that the matrices are similar.

   (c) Find two 4×4 matrices over the complex numbers, which are not similar but have the same characteristic polynomial and the same minimal polynomial.

6. (a) How many non-isomorphic groups of order 99 are there? Fully justify your answer.

   (b) Find six non-isomorphic groups of order 81, and justify your answer.

7. Suppose R is a unitary ring and M is a nontrivial finitely generated left R-module. Prove that M has a nontrivial quotient N that is *simple*, i.e., such that (0) and N are the only submodules of N.

8. Let $K = F(t, u)$ be the field of rational functions in the indeterminates t, u over a field F of characteristic 2. And suppose L is the splitting field over K of the polynomial $(X^2 - t)(X^2 - u)$.

   (a) Prove that the degree $[L : K] = 4$.

   (b) Prove that L is not a simple extension of K. That is, show $L \neq K(\alpha)$ for any α in L.

   (c) Show that there are infinitely many fields between K and L.