The eight questions have equal weight. Attempt all of the questions.

1. Let $T : V \to V$ be a linear operator on an $n$-dimensional, complex vector space. Prove that there exists a tower of $T$-invariant subspaces

$$(0) \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n = V$$

such that $\dim V_j = j$ for every $j$. Deduce from this that $V$ contains a basis such that the matrix of $T$ using this basis is upper triangular. Do not invoke the Jordan canonical form theorem in your proof.

2. (a) Explain the class equation for a finite group, and illustrate it in the case of the group $S_4$ of permutations on 4 letters.

(b) Prove that every group of order $p^2$ is abelian.

3. (a) Show that $1 + \sqrt{-5}$ is irreducible but not prime in the ring $\mathbb{Z} [\sqrt{-5}]$.

(b) Prove that the ideal $(2, 1 + \sqrt{-5})$ is maximal in $\mathbb{Z} [\sqrt{-5}]$.

4. (a) Let $K$ be a Galois extension of degree 12 over a field $F$. Prove that there is an intermediate field $E$ between $F$ and $K$ such that $E$ has degree 3 over $F$.

(b) If $\zeta$ is a primitive 13th root of unity, how many subfields does the field $\mathbb{Q}(\zeta)$ contain? Justify your answer.

(c) If $f(X)$ in $\mathbb{Q}[X]$ is an irreducible polynomial of degree $n \geq 2$, with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{C}$, show that $\sum_{j=1}^{n} \frac{1}{\alpha_j^2} \in \mathbb{Q}$. 
5. (a) Let $V$ be the vector space $P_3(\mathbb{R})$ of polynomials of degree $\leq 3$ with coefficients in $\mathbb{R}$. Let $T : V \to V$ be the linear operator $T(p(x)) = xp''(x)$. Find the Jordan canonical form of $T$ and a corresponding basis for $V$.

(b) If two $3 \times 3$ matrices over the complex numbers have the same characteristic and the same minimal polynomial, prove that the matrices are similar.

(c) Find two $4 \times 4$ matrices over the complex numbers, which are not similar but have the same characteristic polynomial and the same minimal polynomial.

6. (a) How many non-isomorphic groups of order 99 are there? Fully justify your answer.

(b) Find six non-isomorphic groups of order 81, and justify your answer.

7. Suppose $R$ is a unitary ring and $M$ is a nontrivial finitely generated left $R$-module. Prove that $M$ has a nontrivial quotient $N$ that is simple, i.e., such that $(0)$ and $N$ are the only submodules of $N$.

8. Let $K = F(t, u)$ be the field of rational functions in the indeterminates $t, u$ over a field $F$ of characteristic 2. And suppose $L$ is the splitting field over $K$ of the polynomial $(X^2 - t)(X^2 - u)$.

(a) Prove that the degree $[L : K] = 4$.

(b) Prove that $L$ is not a simple extension of $K$. That is, show $L \neq K(\alpha)$ for any $\alpha$ in $L$.

(c) Show that there are infinitely many fields between $K$ and $L$. 

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