ALGEBRA Comprehensive Examination

Wednesday, 22 January 2014

Instructions: Attempt all nine questions. You must show all of your reasoning. All rings can be assumed to have a multiplicative identity.

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[1] Show that a group G of order $2014 = 2 \cdot 19 \cdot 53$ is solvable.

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[2] Let K be an algebraically closed field of characteristic 2 and let A be the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

in $M_4(K)$. Give the Jordan form of A.

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[3] Let I be the ideal $(x^3 - 2x^2 + 3x - 6, x^2 + x)$ of $\mathbb{Z}[x]$. Find a nonzero constant polynomial in I.

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[4] Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial whose Galois group is isomorphic to the quaternion group Q. Prove that $\deg(f(x)) = 8$.

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[5] Let $0 \to V_1 \to V_2 \to \ldots \to V_n \to 0$ be an exact sequence of finite-dimensional vector spaces over a field F. Prove that $\sum_{i=1}^{n} (-1)^i \dim V_i = 0$.

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[6] Let R be a Noetherian ring and let M be a finitely generated R-module. Suppose that $f : M \to M$ is an R-module homomorphism. Show that if f is surjective then f is injective.

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[7] Give the degrees of the spitting fields over the rationals of the following polynomials:

(a) $x^3 - 1$,
(b) $x^6 - 1$,
(c) $x^3 + 3$.

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[8] Let G be a simple group, and let G act nontrivially on a finite set X, with $n \geq 3$ elements. Prove that G is finite, and that $\#G$ divides evenly into $n!/2$.

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[9] Let R be a finite commutative ring.

(a) Show that if the units group of R has odd order then R has characteristic 2.
(b) Show that if R has characteristic 2 then every nonzero ideal of R has size 2^j for some $j \geq 1$.
(c) Show that if R has characteristic 2 and the Jacobson radical, $J(R)$, of R is nonzero then the set $\{1 + x \colon x \in J(R)\}$ is a subgroup of the units group of order 2^m for some $m \geq 1$.
(d) Show that the group of units of R cannot be isomorphic to $\mathbb{Z}_5$.
