University of Waterloo
Department of Pure Mathematics
Algebra Comprehensive Examination
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Prepared by Ruxandra Moraru and Matthew Satriano

REMINDER: The Comprehensive Exams will be static open book, in the sense that consultation of all static pre-downloaded materials will be allowed. Examples of allowable aids include standard texts, and personal study notes. However, no other aids are permitted. In particular, discussing (over any medium) the exam with any person during the availability period, or using the internet or any other non-static tool to search questions or concepts, is not allowed.

Your submitted solutions must reflect your own understanding of the concepts being tested, in your own words.

When you submit a Comprehensive Exam, you are agreeing to the following Academic Integrity Statement:

INTEGRITY STATEMENT

I declare the following statements to be true:

1. The work I submit here is entirely my own.
2. I have not used any unauthorized aids.
3. I have not discussed and will not discuss the contents of this examination with anyone until after the submission deadline.
4. I am aware that misconduct related to examinations can result in significant penalties, including failing the examination and suspension (this is covered in Policy 71: `https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-71`)

Instructions: Answer all questions; they are equally weighted.

Linear algebra

(1) Let A be a complex $n \times n$ matrix.

(a) Prove that if all the eigenvalues of A are real, then A is similar to a real matrix.

(b) Classify up to similarity all the matrices A such that $A^n = I$.

(2) (a) Let V be the vector space $P_2(\mathbb{R})$ of all polynomials of degree ≤ 2 with coefficients in $\mathbb{R}$. Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)\ dx, \quad f_2(p) = p'(1), \quad f_3(p) = p(0).$$

Show that $\{f_1, f_2, f_3\}$ is a basis for the dual space V^* by exhibiting the basis for V of which it is the dual.

(b) Let W be the vector space $M_n(F)$ of $n \times n$ matrices over the field F, and let W^* be its dual space. For a fixed element B of W, set $f_B : W \to F, A \mapsto \operatorname{tr}(B^t A)$. Verify that $f_B \in W^*$. Moreover, show that $\phi : W \to W^*, B \mapsto f_B$, is a linear isomorphism.

Ring Theory

(3) Let R be a principal ideal domain.

(a) Prove or disprove that R is Noetherian. In addition, prove or disprove that R is Artinian.

(b) Let I be a nonzero ideal in R. Prove or disprove that the quotient ring R/I is Noetherian. Moreover, prove or disprove that R/I is Artinian.

(4) (a) Let R be the ring of real-valued continuous functions on $(0, 1)$. Let

$$I = \{f \in R : f(1/3) = 0\}.$$

Prove that I is a maximal ideal in R.

(b) Show that a surjective homomorphism from a field onto a ring with more than one element must be an isomorphism.

Group Theory

(5) Prove that there are no simple groups of order 30.

(6) Let G be a finite p-group and let $\{1\} \neq H \lhd G$. Prove that $H \cap Z(G) \neq \{1\}$.

Field Theory

(7) Let K be the splitting field of $f(x) = x^3 + 2x - 4$.

(a) Compute the Galois group $G(K/\mathbb{Q})$.

(b) List all fields L with $\mathbb{Q} \subset L \subset K$ for which L is a Galois extension of $\mathbb{Q}$.

(8) Let q be a prime power. Denote by $\mathbb{F}_{q^m}$ the finite field with q^m elements for any positive integer m. Fix a positive integer r, and fix $\alpha \in \mathbb{F}_{q^r}$. Prove that the trace $T_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\alpha) = 0$ if and only if there exists $\beta \in \mathbb{F}_q$ such that $\alpha = \beta^q - \beta$.