Department of Pure Mathematics

Algebra Comprehensive Examination

January 26, 2012

1pm-4pm, 3 hours at MC5045

Prepared by W. Kuo and R. Willard

Answer eight questions, including at least one from each of the four sections. Part (a), (b), (c), etc, of a question are often but not always related.

Linear Algebra

1. Let A be an $n \times n$ matrix over a field F. Prove the following.
  (a) If A is nilpotent (that is, $A^m = \mathbf{0}$, the zero matrix, for some m), then $\operatorname{Tr} A^r = 0$ for all $r \geq 1$.
  (b) If the characteristic of F is 0, and $\operatorname{Tr} A^r = 0$ for all $r \geq 1$, then A is nilpotent.

2. (a) Let J be an $n \times n$ Jordan block. Show that any matrix that commutes with J is a polynomial in J.
  (b) Let A be a symmetric $n \times n$ matrix such that $A^2 = K + pI$, where K is the $n \times n$ matrix with all entries equal to 1, I is the $n \times n$ identity matrix, and $p \geq 0$ is a real number. What are the possible eigenvalues of A?

Groups

3. (a) Let G be a group and let N be a normal subgroup of index n. Show that $g^n \in N$ for all $g \in G$.
  (b) Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Show that if the order of H is relatively prime to the index of N in G, then $H \subseteq N$.
  (c) Let G be a finite group and let M be a maximal subgroup of G. Show that if M is a normal subgroup of G, then $[G : M]$ is prime.

4. (a) Let G be a group of order $153 = 3^2 \cdot 17$. Prove that G is abelian.
  (b) Determine all finitely generated abelian groups G whose automorphism group is finite.

5. (a) Prove that if G is a finite group containing no subgroup of index 2, then any subgroup of index 3 is normal in G.
  (b) Let H be a group of order 9. Show that $|\mathrm{Aut}(H)| \mid 48$ (Hint: you may assume that the group $\mathrm{GL}_2(\mathbb{F}_3)$ has order 48).

Rings

6. In this problem R is an integral domain. For $x, y \in R$ we write $x|y$ if $y = xz$ for some $z \in R$. We write $x \sim y$ if $y = xu$ for some unit u in R.

(a) Suppose R is Noetherian. Prove that if $a_1, a_2, a_3, \ldots$ are nonzero elements of R with $a_{i+1}|a_i$ for all $i \geq 1$, then there exists N such that $a_m \sim a_n$ for all $m, n \geq N$.

(b) Suppose R is not a PID. Prove that there exists a nonprincipal ideal I of R with the property that for all $a \in R \setminus I$, the ideal $I + (a)$ is principal.

(c) : Show that if R is not a PID, then R has a nonprincipal prime ideal.

7. Let R be the following subring of $M_2(\mathbb{C})$:

$$R = \left\{ \begin{bmatrix} a & r \\ 0 & s \end{bmatrix} : a \in \mathbb{Q} \text{ and } r, s \in \mathbb{C} \right\}.$$

(a) Prove that R is both right Noetherian and right Artinian.

(b) Prove that R has uncountably many left ideals.

(c) Is R is left Noetherian? Is R is left Artinian? Justify your answers.

Fields

8. Let $\zeta_7 = e^{2\pi i/7}$, let $E = \mathbb{Q}(\zeta_7)$, and let G be the Galois group of E over $\mathbb{Q}$.

(a) Show that G is cyclic of order 6.

(b) Show that there are exactly two fields K satisfying $\mathbb{Q} < K < E$.

(c) If ψ is a generator of G, prove that $\psi(\cos \frac{2\pi}{7}) \in \mathbb{R}$ and determine whether $\psi(\cos \frac{2\pi}{7})$ is positive or negative. (Hint: $\cos \frac{2\pi}{7} = \frac{1}{2}(\zeta_7 + \zeta_7^{-1})$.)

9. (a) Let $f(x) = x^4 + x^3 + x^2 + x + 1$ and $F = \mathbb{Z}_2[x]/(f)$, and put $\alpha = x + (f) \in F$.

(i) Prove that F is a finite field. What is the size of F?

(ii) Find a primitive element for F; express your answer as a polynomial in α (e.g., α, $\alpha + 1$, $\alpha^2 + 1$ etc.)

(b) Prove that any two finite fields of the same size are isomorphic.

10. Suppose F is a field, E is an extension field, $G = \mathrm{Aut}_F(E)$ is the group of automorphisms of E fixing F pointwise, H is a *finite* subgroup of G, and K is the fixed field of H.

(a) Prove that E is algebraic over K.

(b) Prove that E is normal and separable over K.

(End of exam)