Department of Pure Mathematics
Algebra Comprehensive Examination
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Instructions: Answer seven of the following eight questions. If you answer all eight, clearly indicate which question you do not want marked.

Linear Algebra

1. Let

\[ A = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}. \]

(a) Show that \( A \) is nilpotent of index 3.
(b) Find the nilpotent matrix \( M \) in Jordan canonical form which is similar to \( A \).

2. Let \( A \) and \( B \) be two \( n \times n \) matrices over \( \mathbb{C} \).

(a) Show that if \( A \) and \( B \) are similar, then they have the same eigenvalues.
(b) Let \( A \) and \( B \) be two idempotent matrices, i.e., \( A^2 = A \) and \( B^2 = B \). Prove that \( A \) and \( B \) are similar if and only if they are equivalent, i.e., there exist invertible matrices \( P \) and \( Q \) such that \( A = PBQ \).

Group Theory

3. Let \( G \) be a finite group of order greater than 2, half of whose elements have order 2, and the other half of the elements form a subgroup of order a power of \( p \), where \( p \) is an odd prime.

(a) Prove that \( G \) is solvable.
(b) Prove that \( G \) is not nilpotent (in fact, it has trivial center).
(c) Give an example of such a group of order 18.

4. (a) State the universal property satisfied by the free group \( F(X) \) generated by a set \( X \).
(b) Let \( G \) be a group with presentation \( \langle u, v : uv^2 = v^2 u \rangle \).
   i. Prove that \( G \) is infinite.
   ii. Prove that \( G \) is non-abelian.
   iii. Prove that \( G \) has nontrivial center.
Ring Theory

5. Let

\[ R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Q} \right\} \]

be a subring of \( M_2(\mathbb{Q}) \).

(a) Describe the Jacobson radical of \( R \).
(b) Prove that \( R \) has precisely two maximal right ideals.
(c) Prove that \( R \) is right Artinian.
(d) Prove that \( R \) has precisely two isomorphism classes of irreducible right modules.

6. Consider the polynomial ring \( \mathbb{Z}[x, y, z] \). Let \( I \) be the ideal generated by \( z^2 - xy \) and let \( R = \mathbb{Z}[x, y, z]/I \).

(a) Prove that \( R \) is an integral domain.
(b) Prove that \( R \) is Noetherian.
(c) Prove that the only units of \( R \) are \( \pm 1 \).
(d) Prove that \( R \) is not a principal ideal domain.

Field and Galois Theory

7. (a) Let \( E \) be a field extension of \( F \). Prove that if \( [E : F] < \infty \), then \( E \) is an algebraic extension of \( F \).

(b) Let \( E \) be a field extension of \( F \). Define

\[ L = \{ \alpha \in E, [F(\alpha) : F] < \infty \}. \]

Prove that \( L \) is a field.
(c) Determine if the converse of (a) is true. Briefly justify your answer.

8. Let \( E \) be the splitting field of \( x^3 - 2 \) over \( F \).

(a) Let \( F = \mathbb{Q} \).
   i. Compute the Galois group \( \text{Gal}_\mathbb{Q}(E) \). Justify your answer.
   ii. Write down the lattice of the corresponding intermediate fields of \( \mathbb{Q} \subseteq E \).

(b) Let \( F = \mathbb{F}_5 \), the finite field of 5 elements.
   i. Compute the Galois group \( \text{Gal}_\mathbb{F}_5(E) \). Justify your answer.
   ii. Write down the lattice of the corresponding intermediate fields of \( \mathbb{F}_5 \subseteq E \).