Algebra Comprehensive Exam: January 29, 2019

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Galois theory

1. Suppose $K = \mathbb{Q}(\sqrt{2+\sqrt{2}})$. Show that $K/\mathbb{Q}$ is Galois and determine its Galois group.
2. Let $p(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 3 with roots a, b, c and let $\Delta := (a-b)(a-c)(b-c)$.
   (a) Show that if the Galois group of $p(x)$ is cyclic of order 3 then $\Delta := (a-b)(a-c)(b-c)$ is a rational number.
   (b) Show that if Δ is rational then the Galois group of $p(x)$ is cyclic of order three.

Linear algebra

1. Let A be an $n \times n$ complex matrix whose characteristic polynomial has no repeated roots. How many $n \times n$ matrices over $\mathbb{C}$ are there that are both similar to and commute with A?
2. Let V be a finite-dimensional complex vector space and let $T : V \to V$ be a linear transformation. Show that $V = W \oplus U$ where W and U are T-invariant subspaces and $T|_U : U \to U$ is nilpotent and $T|_W : W \to W$ is an isomorphism.

Group theory

1. Prove that a group G of order 105 is not simple.
2. (a) Let G be a finite group and let H be a proper subgroup. Show that G is not equal to the union of gHg^{-1} as g ranges over the elements of G.
   (b) Show that it is possible for an infinite group G to be the union of conjugates of proper subgroup. (Hint: Look at $G = \mathrm{GL}_n(\mathbb{C})$ with $n \geq 2$.)

Ring theory

1. (a) Let R be a ring and let $f : R \to R$ be a surjective homomorphism. Show that if the kernel of f is nonzero then

$$(0) \subseteq \ker(f) \subseteq \ker(f \circ f) \subseteq \ker(f \circ f \circ f) \subseteq \cdots$$

   is an ascending chain of ideals of R that does not terminate.
   (b) Let k be a field and let $f : k[x_1, \ldots, x_d] \to k[x_1, \ldots, x_d]$ be a k-algebra homomorphism. Show that if f is surjective then f is injective.
2. Let $R = M_n(\mathbb{Z})$ and let J be a two-sided ideal of R. Show that there is some integer d such that $J = M_n(d\mathbb{Z})$; i.e., the set of matrices whose entries are all multiples of d.