Algebra Comprehensive Exam

January 30, 2013, MC5046, 2:30-5:30pm

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- Attempt all questions.
- You must show all of your reasoning.
- Each of the four sections has roughly equal weight.

Linear Algebra

1. Compute the Jordan canonical form of the 4×4 complex matrix

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

whose characteristic polynomial is $(x-1)(x-2)(x-4)^2$.

2. Prove that there are $n+1$ similarity classes of idempotent $n \times n$ matrices (i.e. matrices E for which $E^2 = E$) over the field of complex numbers.

Group Theory

3. **(a)** Show that no group of order 56 is simple.

   (b) Give a clear statement of any major theorem used in the proof of part (a).

4. Let p be a prime.

   (a) Let C_p denote the cyclic group of order p. Prove that $\mathrm{Aut}(C_p \times C_p)$ has order $(p^2-1)(p^2-p)$.

   (b) Prove that there exists a non-abelian group of order p^3.

5. Let $\mathbb{F} = \langle u, v \rangle$ be the free group on two generators u and v, and N be the normal subgroup generated by $uvu^{-2}v^{-1}$. Let $G = \mathbb{F}/N$.

   (a) Find a homomorphism α from $\mathbb{F}$ onto the infinite cyclic group, for which $N \subseteq \ker \alpha$. Conclude that G is infinite.

   (b) Find a homomorphism β from $\mathbb{F}$ onto the symmetric group S_3, for which $N \subseteq \ker \beta$. Conclude that G is non-abelian.

In the following sections, $\mathbb{Q}$ will always denote the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{Z}$ the ring of integers.

Ring Theory

6. (a) Define what is meant by a *prime* ideal and a *maximal* ideal in a commutative ring R.

   (b) Let d be an integer which is not a perfect square and let $R = \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$. Show that every non-zero prime ideal of R is maximal.

7. Consider the ring

$$T = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix} \times \mathbb{R} := \left\{ \begin{bmatrix} q & r \\ 0 & s \end{bmatrix} : q, r, s \in \mathbb{Q} \right\} \times \mathbb{R}.$$

   (a) Compute the Jacobson radical of T.

   (b) Find all of the maximal (2-sided) ideals of T.

Field Theory

8. (a) Determine the Galois closure of $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$.

   (b) Find the lattice of all subfields of $\mathbb{Q}(\sqrt[4]{2})$.

9. Let L/K and M/K be finite extensions of respective degrees n and m, of a field K, which satisfy $L \cap M = K$.

   (a) Show that if $\gcd(n, m) = 1$, then the composite field LM has degree nm over K.

   (b) Find examples of K, L and M, for which LM does not have degree nm over K.

10. Let F be a field with p^m elements where p is a prime and m is a positive integer, and let K and L be subfields of respective cardinalities p^k and p^ℓ where $0 < k, \ell < m$. Calculate all of the possible cardinalities of $K \cap L$.