Algebra Comprehensive Exam
January 30, 2013, MC5046, 2:30-5:30pm
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- Attempt all questions.
- You must show all of your reasoning.
- Each of the four sections has roughly equal weight.

Linear Algebra

1. Compute the Jordan canonical form of the $4 \times 4$ complex matrix

$$A = \begin{bmatrix}
5 & 4 & 2 & 1 \\
0 & 1 & -1 & -1 \\
-1 & -1 & 3 & 0 \\
1 & 1 & -1 & 2
\end{bmatrix}$$

whose characteristic polynomial is $(x - 1)(x - 2)(x - 4)^2$.

2. Prove that there are $n+1$ similarity classes of idempotent $n \times n$ matrices (i.e. matrices $E$ for which $E^2 = E$) over the field of complex numbers.

Group Theory

3. (a) Show that no group of order 56 is simple.
   
   (b) Give a clear statement of any major theorem used in the proof of part (a).

4. Let $p$ be a prime.
   
   (a) Let $C_p$ denote the cyclic group of order $p$. Prove that $\text{Aut}(C_p \times C_p)$ has order $(p^2 - 1)(p^2 - p)$.
   
   (b) Prove that there exists a non-abelian group of order $p^3$.

5. Let $\mathbb{F} = \langle u, v \rangle$ be the free group on two generators $u$ and $v$, and $N$ be the normal subgroup generated by $uvu^{-2}v^{-1}$. Let $G = \mathbb{F}/N$.
   
   (a) Find a homomorphism $\alpha$ from $\mathbb{F}$ onto the infinite cyclic group, for which $N \subseteq \ker \alpha$. Conclude that $G$ is infinite.
   
   (b) Find a homomorphism $\beta$ from $\mathbb{F}$ onto the symmetric group $S_3$, for which $N \subseteq \ker \beta$. Conclude that $G$ is non-abelian.
In the following sections, \( \mathbb{Q} \) will always denote the field of rational numbers, \( \mathbb{R} \) the field of real numbers, and \( \mathbb{Z} \) the ring of integers.

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**Ring Theory**

6. (a) Define what is meant by a *prime* ideal and a *maximal* ideal in a commutative ring \( R \).

(b) Let \( d \) be an integer which is not a perfect square and let \( R = \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\} \). Show that every non-zero prime ideal of \( R \) is maximal.

7. Consider the ring

\[
T = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix} \times \mathbb{R} := \left\{ \begin{bmatrix} q & r \\ 0 & s \end{bmatrix} : q, r, s \in \mathbb{Q} \right\} \times \mathbb{R}.
\]

(a) Compute the Jacobson radical of \( T \).

(b) Find all of the maximal (2-sided) ideals of \( T \).

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**Field Theory**

8. (a) Determine the Galois closure of \( \mathbb{Q}(\sqrt{2})/\mathbb{Q} \).

(b) Find the lattice of all subfields of \( \mathbb{Q}(\sqrt{2}) \).

9. Let \( L/K \) and \( M/K \) be finite extensions of respective degrees \( n \) and \( m \), of a field \( K \), which satisfy \( L \cap M = K \).

(a) Show that if gcd\((n, m) = 1\), then the composite field \( LM \) has degree nm over \( K \).

(b) Find examples of \( K, L \) and \( M \), for which \( LM \) does not have degree \( nm \) over \( K \).

10. Let \( F \) be a field with \( p^m \) elements where \( p \) is a prime and \( m \) is a positive integer, and let \( K \) and \( L \) be subfields of respective cardinalities \( p^k \) and \( p^\ell \) where \( 0 < k, \ell < m \). Calculate all of the possible cardinalities of \( K \cap L \).

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