University of Waterloo
Department of Pure Mathematics
Analysis and Topology Comprehensive Examination
1:00 p.m.–4:00 p.m., Wednesday May 13, 2015

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Instructions: Answer ALL questions in Part I.
In Part II, do ONE problem from each section.
Questions in Part I are marked out of 5; questions in Part II are marked out of 10.

PART I

Do all questions. Provide **brief but complete** answers **with explanations**.

I 1. Let $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$. Does $\displaystyle\int_0^1 f(x)\,dx$ exist as a Riemann integral? Does it exist as a Lebesgue integral?

I 2. Suppose that $f(x)$ is a continuous complex valued function on $[0,\infty)$ and $\lim_{x\to\infty} f(x) = 0$. Prove that f can be uniformly approximated on $[0,\infty)$ by a sequence of functions of the form $q_n(x) = \displaystyle\sum_{k=1}^{n} a_k e^{-kx}$ where $a_k \in \mathbb{C}$.

I 3. Suppose that f is an L^2 function on the unit disk $\mathbb{D}$ in $\mathbb{C}$ with respect to planar Lebesgue measure. Suppose further that $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. Prove that $\|f\|_2^2 = \displaystyle\sum_{n=0}^{\infty} \frac{\pi |a_n|^2}{n+1}$.

I 4. Find an analytic function $f(z)$ defined on $\{z \in \mathbb{C} : x > 0\}$, where $z = x + iy$ and $x, y \in \mathbb{R}$, whose real part is $u(x,y) = \log(x^2 + y^2)$.

I 5. How many roots (counting multiplicity) does $f(z) = z^7 + 5z^3 - z - 2$ have in the open unit disc?

I 6. Show that every infinite set is the disjoint union of countably infinite subsets.

I 7. Let $\lfloor y \rfloor$ be the integer part of y, let $A_n = \{x \in [0,1] \mid \lfloor 2^n x \rfloor \text{ is even}\}$, and let $g_n = \chi_{A_n}$ be the characteristic function of A_n. Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 f g_n\,dx = \frac{1}{2}\int_0^1 f\,dx$ for all $f \in L^1(0,1)$.

I 8. Let X be a topological space and $\sim$ be an equivalence relation on X. Let $X/{\sim}$ be the set of equivalence classes and $\pi \colon X \to (X/{\sim})$ be the projection. Define the quotient topology on $X/{\sim}$ and prove that $f \colon (X/{\sim}) \to Y$ is continuous if and only if $\hat{f} := f \circ \pi$ is continuous.

Part II

Do one problem from each section. If you attempt both problems in a section, then you must clearly indicate which one you want marked. Otherwise only the first one encountered by the grader will be marked.

Basic Real Analysis. Answer ONE question.

A1. Let $a > 0$ and define $f(t) = e^{at}$ for $-\pi \leq t \leq \pi$.

(a) Find the Fourier series of f.

(b) Use a computation of $\|f\|_2$ to evaluate the sum $\dfrac{1}{a^2} + 2\displaystyle\sum_{n\geq 1} \frac{1}{a^2+n^2}$.

A2. Prove that $[0,1]$ is not the *disjoint* union of a countably infinite collection of non-empty closed sets A_n. Hint: consider $X = [0,1] \setminus \bigcup_{n\geq 1} \operatorname{int}(A_n)$.

Complex Analysis. Answer ONE question.

B1. Let Ω be a simply connected domain properly contained in $\mathbb{C}$, and let $z_0 \in \Omega$. Suppose that f is holomorphic on Ω, $f(\Omega) \subset \Omega$ and $f(z_0) = z_0$.

(a) Prove that $|f'(z_0)| \leq 1$.

(b) What more can be said when $|f'(z_0)| = 1$?

B2. (a) For which real a does $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx$ make sense as an improper Riemann integral?

(b) Evaluate this integral for those values of a.

(c) What meaning can be given to the formula you obtained in (b) for other values of a?

MEASURE THEORY. ANSWER ONE QUESTION.

C1. Find $\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\, dx$. Justify your arguments carefully.

C2. Let $I = [0,1]$ with Lebesgue measure, and let $p \in [1,\infty)$. Consider a sequence $f_k \in L^p(I)$ with $\|f_k\|_p \le 1$. Suppose that $f(x) = \lim_{k\to\infty} f_k(x)$ exists for almost every x.

Does f belong to $L^p(I)$? Prove it or give a counterexample.

TOPOLOGY AND SET THEORY. ANSWER ONE QUESTION.

D1. Put the lexicographic order on $X = [0,1]^2$ defined by

$$(x_1, y_1) < (x_2, y_2) \quad \text{if} \quad x_1 < x_2 \quad \text{or} \quad x_1 = x_2 \text{ and } y_1 < y_2.$$

Let $\mathcal{T}$ be the order topology generated by the sets

$$\{(x,y) : (x_1,y_1) < (x,y)\} \quad \text{and} \quad \{(x,y) : (x,y) < (x_2,y_2)\}.$$

(a) Show that every subset of X has a least upper bound.

(b) What is the induced topology on $Y = \{(x,y) : y = \frac{1}{2}\}$?

(c) What is the closure of Y?

D2. Let A be an infinite set. A *chain* in the power set $\mathcal{P}(A)$ is a subset of $\mathcal{P}(A)$ which is totally ordered by inclusion.

(a) Prove that $\mathcal{P}(A)$ contains maximal chains.

(b) Prove that there are maximal chains of cardinality $|A|$.

(c) Prove that $\mathcal{P}(\mathbb{N})$ contains maximal chains of cardinality $2^{|\mathbb{N}|}$.