Analysis and Topology Comprehensive Exam

May 27, 2019; 1:00pm-4:00pm in MC 5417
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Instructions. Attempt all questions. Show all your work. The questions are not of equal difficulty.

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(1) Parts (a) and (b) of the following question are *not* related.

(a) Let G be a non-trivial subgroup of $(\mathbb{R}, +)$. Prove that either G is dense in $\mathbb{R}$, or there exists $\alpha > 0$ such that $G = \alpha\mathbb{Z}$.

(b) (i) Let $(b_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, and suppose that $\beta := \lim_{n\to\infty} b_n$ exists in $\mathbb{R}$. Prove that

$$\lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=1}^{N} b_n \right) = \beta.$$

(ii) Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} a_n$ converges to some real number α. Prove that

$$\lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=1}^{N} n a_n \right) = 0.$$

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(2) Evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)} dx.$$

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(3) These questions are similar but they are *not* directly related to each other. Let G be a group of $n \times n$ matrices over either $\mathbb{R}$ or $\mathbb{C}$. We give G the subspace topology as a subset of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ or $M_{n\times n}(\mathbb{C}) \cong \mathbb{C}^{n^2}$, respectively.

(a) Let $G = \mathrm{GL}(n, \mathbb{R})$ be the group of invertible $n \times n$ real matrices. Show that G is *not connected.*

(b) Let $H = \mathrm{SU}(n)$ be the group of $n \times n$ *special unitary* complex matrices. Show that H is both compact and connected.

(c) Show that $\mathrm{SU}(2)$ is homeomorphic to $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} x_i^2 = 1\}$.

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(4) Consider the set $[0, 1]$ equipped with Lebesgue measure μ. Let $0 \neq f \in L^{\infty}([0, 1], \mu)$, and let $\alpha_n := \int_{[0,1]} |f|^n \, d\mu$ for all $n \geq 1$. Prove that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_{\infty}.$$

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(5) Let A be a complex $n \times n$ matrix. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A.

(a) If $|z| > \max_{k=1}^{n} |\lambda_k|$, express $(zI - A)^{-1}$ as a convergent power series.

(b) Let $p(z)$ be a polynomial. Let $R > \max_{k=1}^{n} |\lambda_k|$ and let $\gamma(t) = Re^{2\pi i t}$ for $t \in [0, 1]$. Prove that

$$p(A) = \frac{1}{2\pi i} \int_{\gamma} p(z)(zI - A)^{-1} dz,$$

where the integral of a matrix of functions is defined by $(\int_{\gamma} B(z)dz)_{ij} = \int_{\gamma} B(z)_{ij} dz$.

(c) Let $q(z) = \det(zI - A)$ be the characteristic polynomial of A. Use part [b] to prove the Cayley-Hamilton theorem, namely that $q(A) = 0$. *Hint: You will need to use the formula* $B\,\mathrm{adj}(B) = (\det B)I$ *where* $\mathrm{adj}\,B$ *is the transpose of the matrix of cofactors of* B.

(6) Let $(V, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, and let D be a linear operator on V such that $D^2 = 0$. Thus $\operatorname{im} D$ is a subspace of $\ker D$. Let H be the quotient space $\ker D / \operatorname{im} D$. Let D^* denote the Hilbert space adjoint of D. Let $[v] = v + \operatorname{im} D$ be an equivalence class in H.

(a) Suppose there exists a representative w of $[v]$ such that $D^* w = 0$. Prove that any representative w' of $[v]$ with $w' \neq w$ satisfies $\|w'\| > \|w\|$.

(b) Suppose there exists a representative w of $[v]$ with *minimal norm.* Prove that $D^* w = 0$.

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(7) (a) Give a precise statement of Rouché's Theorem.

(b) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function which is *not* a polynomial. For each $N \geq 1$, set

$$s_N(z) := \sum_{k=0}^{N} a_k z^k, \qquad z \in \mathbb{C}.$$

Prove that for each $R > 0$, there exists an integer $M_0 \geq 1$ such that for all $N \geq M_0$, the function $s_N(z)$ has at least one root outside of the disc $D := \{z \in \mathbb{C} : |z| \leq R\}$.

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(8) Let μ denote Lebesgue measure on $\mathbb{R}$. Let $(f_n)_n$ and f be real-valued Lebesgue measurable functions on $[0, 1]$. Recall that we say that $(f_n)_n$ converges to f *in measure* if for all $\varepsilon > 0$,

$$\lim_{n \to \infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) = 0.$$

Prove that the following two conditions are equivalent:

(i) $(f_n)_n$ converges to f in measure on $[0, 1]$.

(ii)

$$\lim_{n \to \infty} \int_{[0,1]} \frac{|f_n - f|}{1 + |f_n - f|} \, d\mu = 0.$$

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(9) Let U be an open set in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be a C^k mapping, where $k \geq 1$.

(a) Give a precise statement of the *inverse function theorem*, including any additional hypotheses that are required. Do not prove this theorem.

(b) Suppose that the differential $(Df)_x$ of f at x is *invertible* for all $x \in U$. Use the inverse function theorem to prove that f is an *open mapping.* That is, if $W \subseteq U$ is open in $\mathbb{R}^n$, prove that $f(W)$ is open in $\mathbb{R}^n$.

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(10) Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane. Let $\mathcal{A}$ be the set of all polynomial functions on $\mathbb{C}$ restricted to S^1. That is,

$$\mathcal{A} = \big\{ f \colon S^1 \to \mathbb{C}; f(e^{i\theta}) = \sum_{k=0}^{n} c_k e^{ik\theta}, c_k \in \mathbb{C} \big\}.$$

(a) Prove that $\mathcal{A}$ is an algebra of functions that separates points in S^1 and includes constant functions.

(b) Either prove that the uniform closure of $\mathcal{A}$ is all of $\mathcal{C}(S^1)$, or give an example (with justification) to show that it is not.
